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DAL PROCESSO DI POISSON AL PROCESSO PSEUDO-POISSONIANO: ASPETTI PROBABILISTICI E STATISTICI

Riassunto

Da una decina d'anni è ripreso l'interesse dei ricercatori per i processi di P. Lévy, considerati tanto dal punto di vista teorico che da quello applicativo. Tra le applicazioni una posizione preminente spetta alla Finanza matematica: uno sviluppo moderno delle antiche discipline Matematica finanziaria e Matematica attuariale. L'attuale Seminario riguarda una modalità di presentazione dei processi stocastici di Lévy che li mette in relazione con i ben noti processi di passeggiata aleatoria e con i processi di Markov – Feller. Gli aspetti strutturali riguardano in particolare due teoremi di approssimazione dei processi di Lévy e dei processi di Markov – Feller: il primo teorema risale ai lavori di B. de Finetti del biennio 1929 – 1930 e sostanzialmente coincide con un primo teorema di rappresentazione per le distribuzioni infinitamente divisibili. Il Seminario affronta anche alcuni problemi di inferenza statistica su un particolare tipo di processi di Lévy, i processi di Poisson composto; vengono considerati sia il noto approccio inferenziale classico che quello bayesiano.

DE FINETTI - LEVY STOCHASTIC PROCESSES

{Z(t)} has **stationary and independent increments** Z(t) – Z(s) ,

{Z(t)} $\rightarrow$ $\lim_{s\to t}\Pr\{|Z(s)-Z(t)|>\varepsilon\}=0$ and Z(0) = 0.

Some examples:

- the **Wiener** process $\begin{cases} Gaussian \quad pr.s \quad with \quad cont. \ trajectories \\ E[Z(t)]\equiv 0, \quad Cov[Z(s),Z(t)]=\min(s,t) \end{cases}$
- the **Poisson** process: $P\left[Z(t)=n\right]=\exp\{-\lambda.t\}.\dfrac{(\lambda.t)^n}{n!}$
- the **compound-Poisson** pr.: $Z(t)=\sum_{j=1}^{N(t)}U_j \quad \begin{cases} N(t)\approx Poisson(\lambda t) \\ U_j / N(t)=n \quad \approx i.i.d. \end{cases}$
- the **Gamma** process; **α – stable** process; etc.

As they have stationary and independent increments :

1) Z(t) ~ **infinitely divisible distribution**: $\forall n, \quad \varphi_{Z(t)}(\xi)=\left[\varphi_n(\xi)\right]^n$;

2) $\varphi_{Z(t+s)}(\xi)=\varphi_{Z(t)}(\xi).\varphi_{Z(s)}(\xi) \quad \Rightarrow \quad \varphi_{Z(t)}(\xi)=\exp\{t.\psi(\xi)\}$;

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3) $\varphi_{Z(t)}(\xi)=E\left[e^{i\xi Z(t)}\right]=\exp\left\{ibt\xi-\dfrac{ct\xi^2}{2}+t.\int_R\left[e^{i\xi x}-1-i\xi x.I_{(-1,1)}(x)\right]\nu(dx)\right\}$

if $\nu\left(\{0\}\right)=0 \quad and \quad \int_R\left(|x|^2\wedge 1\right)\nu(dx)<\infty$; $\quad \varphi_{Z(t)}(\xi)\Leftrightarrow(b,c,\nu)$

4) $Z(t)=m(t)+Z_c(t)+Z_j(t), \quad Z_c(t) \ \coprod \ Z_j(t)$.

ALCUNI RIFERIMENTI BIBLIOGRAFICI DEI PRIMI CONTRIBUTI SUI PROCESSI CON INCREMENTI UGUALMENTE DISTRIBUITI E INDIPENDENTI

1928) B. de Finetti: Funzione caratteristica di un fenomeno aleatorio. (Atti Congr. Int. dei Matematici, Bologna, 1928)

1929) B. de Finetti: Sulle funzioni ad incremento aleatorio. (Rend. Acc. Naz. Lincei)

1929) B. de Finetti: Integrazione delle funzioni a incremento aleatorio. (Rend. Acc. Naz. Lincei)

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1930) B. de Finetti: Le funzioni caratteristiche di legge istantanea. (Rend. Acc. Naz. Lincei)

1931) B. de Finetti: Le funzioni caratteristiche di legge istantanea dotate di valori eccezionali. (Rend. Acc. Naz. Lincei)

1931) B. de Finetti: Le leggi differenziali e la rinunzia al determinismo. (Univ. di Roma)

1932) A.N. Kolmogorov: Sulla forma generale di un processo stocastico omogeneo (Un problema di Bruno de Finetti). (Rend. Acc. Naz. Lincei)

1934-35) P. Lévy: Sur les intégrales dont les éléments sont des variables aléatoires indépendantes. (Ann. Scuola Norm. Sup. Pisa)

1937-1942) B. de Finetti (“Funzioni aleatorie” , Congresso UMI, 1937)
W. Feller,
A.Ya. Khintchine,
K. Ito and others.

Let $W_n = \sum_{j=1}^{n} T_j$, $where \quad T_j \approx i.i.d. - Neg.\exp(\lambda)$ be a **random walk process**.

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1) Poisson process {N(t); $t \geq 0$**}**

a) $N(0) = 0 \quad and \quad N(t) = n \quad iff \quad W_n \leq t < W_{n+1} \quad \rightarrow$

$\rightarrow \quad P\{N(t) = n\} = e^{-\lambda t}.(\lambda t)^n / n! \qquad \Phi_{N(t)}(\xi) = \exp\{\lambda t(e^{i\xi} - 1)\}$;

b) It is a <u>Lévy process</u> characterized by $E[N(t)] = \lambda t \quad and \quad \nu_t(dx) = \lambda t. I_{\{1\}}(dx)$

2) Compound-Poisson process {X(t);$t \geq 0$**}:** it is a random walk $S_k =$ $= \sum_{j=1}^{k} Y_j$ with transition times given by the random walk process $\{W_n\}$.

a) $X(t) = \sum_{j=1}^{N(t)} Y_j = \sum_{n \geq 1} S_n.|N(t) = n|$, $where \quad Y_j / N(t) = n \approx i.i.d. - F_Y(dy) \quad \rightarrow$

$\rightarrow \quad F_{X(t)}(x) = e^{-\lambda t}.\sum_{n \geq 0} \frac{(\lambda t)^n}{n!} F_Y^{n*}(x) \quad , \quad \Phi_{X(t)}(\xi) = \exp\{\lambda t[\Phi_Y(\xi) - 1]\}$;

b) It is a <u>Lévy process</u> characterized by $E[X(t)] = \lambda t.E(Y), \quad \nu_t(dx) = \lambda t. F_Y(dx)$

Approximation theorem (1)

3) Pseudo-Poisson process X(t) = $\sum S_n.|N(t) = n|$**:** it is a Markov chain $\{S_k\}$ with transition times given by the random walk process $\{W_n\}$.

$$p_{ij}(t) = e^{-\lambda t}.\sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!}.p_{ij}^n \quad where \quad \left[p_{ij}^n\right] = P^n \ ;$$

$$P\{X(t+s) \in \Gamma / X(s) = x\} = Q_t(x, \Gamma) = e^{-\lambda t}.\sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} K^{(n)}(x, \Gamma),$$

$$K^{(n)}(x, \Gamma) = \int_R K(x, dy).K^{(n-1)}(y, \Gamma).$$

Approximation theorem (2)

1) Exponential formula of semi-group of contractions theory

$$Q^*(t) = e^{-\lambda.t}.\sum_{n=0}^{\infty} \frac{(\lambda.t)^n}{n!} (T^*)^n = e^{\lambda.t(T^* - I)}; \qquad (T^*)^{(n)} u(x) = \int_R K^{(n)}(x, dy).u(y).$$

COMPOUND-POISSON process $X(t)\ \begin{cases} N(t) \approx Poisson \quad process \quad (\lambda t) \\ random \quad walk \quad S_k = \sum_1^k Y_j \\ \forall n, \quad Y_j / N(t) = n \approx i.i.d. \end{cases}$

An example:

$$f_Y(y) \approx \Gamma(1,\beta) \Rightarrow f_{X(t)}(x) = \sum_{n=0}^{\infty} P\left[N(t)=n\right].f_{X(t)/N(t)=n}(x) = \sum_{n=0}^{\infty}\left(\frac{e^{-\lambda t}.(\lambda.t)^n}{n!}\right).f_Y^{n*}(x),$$

where $f_Y^{n*}(x) \approx \Gamma(n,\beta)$, and this implies $E[X(t)] = E[N(t)].E(Y_j) = \lambda.t/\beta$

and $Var\left[X(t)\right] = 2\lambda t/\beta^2$.

Maximum likelihood inferential procedure about the parameters λ and β

$$\ell(\lambda,\beta/data) \propto (\lambda.\beta)^{\sum_{k=1}^{m} n_k}.\exp\left\{-m.\lambda - \beta.\sum_{k=1}^{m}\sum_{j=1}^{n_k} y_{jk}\right\} \quad , \quad \text{where}$$

m $\approx$ number of the unitary periods of observation,

$n_k \approx$ number of arrivals in the k-th period of observation,

$\sum_{j=1}^{n_k} y_{jk} \approx$ cumulative effect in the k-th period of observation.

Maximum likelihood estimates $\begin{cases} \hat{\lambda} = \sum_{k=1}^{m} n_k / m \\ \hat{\beta} = \sum_{k=1}^{m} n_k / \sum_{k=1}^{m}\sum_{j=1}^{n_k} y_{jk} \end{cases}$

PSEUDO-POISSON process X(t) $\begin{cases} discrete-time \quad Markov \quad chain \quad (K) \\ Poisson \quad process \quad (\alpha t) \end{cases}$

$$P\{X(t+s)\in\Gamma / X(s)=x\} = Q_t(x,\Gamma) = e^{-\alpha t}.\sum_{n=0}^{\infty}\frac{(\alpha t)^n}{n!}K^{(n)}(x,\Gamma),$$

$$K^{(n)}(x,\Gamma) = \int_R K(x,dy).K^{(n-1)}(y,\Gamma), \quad n>1; \qquad K^{(1)}(x,\Gamma) = K(x,\Gamma) \; .$$

$$Q_{t+\tau}(x,\Gamma) = \int_R Q_t(x,dy)Q_\tau(y,\Gamma)$$

As an example we may consider a particle travelling at uniform speed through homogeneous matter occasionally scoring a collision. Each collision produces a change of energy regulated by a stochastic kernel K. The transition probabilities have the above expression if the number of collisions obeys a Poisson process. It is usually assumed that the fraction of energy lost at each collision is independent of the initial amount. Let us first assume that the fraction of energy lost is uniformly distributed:

$$k(x,y)\equiv 1/x, \quad x>0, \quad \Rightarrow \quad k^{(n)}(x,y) = \frac{1}{\Gamma(n).x}\left(\ln\frac{x}{y}\right)^{n-1}, \quad y\in(0,x);$$

$$r=\frac{y}{x} \approx f_n(r) = \frac{1}{\Gamma(n)}\left(\ln\frac{1}{r}\right)^{n-1}, \quad r\in(0,1);$$

$$q_t(x,y) = \frac{\alpha.t}{x.e^{\alpha.t}}\sum_{n=0}^{\infty}\frac{[\alpha.t.\ln(x/y)]^n}{n! \;\; \Gamma(n+2)} \quad \leftrightarrow \quad f_t(r) = \frac{\alpha.t}{e^{\alpha.t}}\sum_{n=0}^{\infty}\frac{[\alpha.t.\ln(1/r)]^n}{n! \;\; \Gamma(n+2)} \; .$$

More generally: $k(x,y)\equiv \lambda.y^{\lambda-1}/x, \quad \Rightarrow \quad k^{(n)}(x,y) = \frac{\lambda^n.y^{\lambda-1}}{\Gamma(n).x^\lambda}\left(\ln\frac{x}{y}\right)^{n-1};$

$$q_t(x,y) = \frac{e^{-\alpha.t}.y^{\lambda-1}.\sqrt{\lambda.\alpha.t}}{x^\lambda.\sqrt{\ln(x/y)}}.I_1\left(2.\sqrt{\lambda.\alpha.t.\ln(x/y)}\right) \; ; \; I_1(x) = \sum_{n=0}^{\infty}\frac{(x/2)^{2n+1}}{n!\Gamma(n+2)}, \quad x\in R.$$

Theoretical hypothesis: if the initial energy may be considered infinitely large, then the total energy loss in [0,t], X(t), has a compound-Poisson distribution.

APPROXIMATION THEOREM (1)

Lemma: there is a one-to-one correspondence between Lévy processes $\{X(t)\}$ and infinitely divisible distributions $\{\mu\}$ through $F_{X(1)}(dx) = \mu(dx)$. The process corresponding to μ is unique up to identity in law.

Theorem: every infinitely divisible distribution (Lévy process) is the limit of a sequence of compound-Poisson distributions (processes).

<u>A short hint</u> (B. de Finetti – Teoria delle Probabilità, II vol., p. 444):

1) let $\Phi(\xi)$ be the characteristic function of an infinitely divisible distribution and $\nu(dx)$ the Lévy measure of the Lévy process corresponding to $\Phi(\xi) \rightarrow \Phi^{1/n}(\xi)$ is a characteristic function for each n;

2) $\exp\{n.[\Phi^{1/n}(\xi)-1]\}$ is the characteristic function of a compound-Poisson distribution for each n;

3) $\exp\{n.[\Phi^{1/n}(\xi)-1]\} \xrightarrow{n\to\infty} \Phi(\xi)$ as $n.\left(x^{1/n}-1\right) \xrightarrow{n\to\infty} \ln x$.

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4) $\exp\{n.[\Phi^{1/n}(\xi)-1]\} = \exp\left\{\int_R \left(e^{i\xi x}-1\right) \; n.F_{1/n}(dx)\right\}$ and it can be proved that

$$\forall \;\; bounded, continuous \;\; f(x): \;\; \int_R f(x) \;\; n.F_{1/n}(dx) \rightarrow \int_R f(x)\nu(dx).$$

Theorem (W. Feller): The following classes of prob. distr.s are identical

1) Infinitely divisible distributions
2) Distributions of the increments in Lévy processes
3) Limits of sequences of compound-Poisson distributions
4) Limits of sequences of infinitely divisible distributions
5) Limit distributions of row sums in triangular arrays $\{X_{k,n}\}$ where the variables $X_{k,n}$ of the n-th row are i.i.d.

CONTINUOUS SEMI-GROUPS OF TRANSITION OPERATORS

Continuous-time Markov process $\rightarrow P\{X(t+s)\in\Gamma / X(s)=x\} = D_t(x,\Gamma)$

$$D_{t+\tau}(x,\Gamma) = \int_R D_t(x,dy) D_\tau(y,\Gamma) \, ;$$

$V \approx$ real valued bounded continuous functions with $\|u(x)\| = \sup |u(x)|$:

$$V \xrightarrow{\Delta(t)} V : \;\; \Delta(t)u(x) = \int_R u(y) D_t(x,dy) = E\left\{u\left[X(t+\tau)\right] / X(\tau)=x\right\}$$

Continuous semi-group of contractions: $\{\Delta(t); t\geq 0\}\begin{cases} \Delta(0) = \Im \\ \Delta(s+t) = \Delta(s)\Delta(t) \\ \|\Delta(t)u\| \leq \|u\| \\ \Delta(t) \xrightarrow{t\downarrow 0} \Im \end{cases}$

Generator A of $\{\Delta(t); t\geq 0\}$: $Au(x) = \lim_{t\downarrow 0} \dfrac{\Delta(t)u(x) - u(x)}{t}$

a) bounded A $\rightarrow$ $\Delta(t)u(x) = \exp\left[t.A\right]u(x) = \sum_{n=0}^{\infty} \dfrac{t^n . A^n u(x)}{n!}$

Some examples $\begin{cases} Compound - Poisson \quad process \\ Pseudo - Poisson \quad process \end{cases}$

b) unbounded A $\rightarrow$ it does not exists the exponential representation

as for a general Feller – Markov process, but

$$\Delta_h(t) = e^{-t/h} . \sum_{n=0}^{\infty} \frac{(t/h)^n}{n!} \Delta(nh) = \exp\left\{\frac{t}{h}\left[\Delta_h - \Im\right]\right\} \xrightarrow{h\rightarrow 0^+} \Delta(t)$$

Transition operator of the pseudo-Poisson process

APPROXIMATION THEOREMS

1) $\left\{X(t); t \geq 0\right\} \approx$ *Lévy process with* $E\left[e^{i\xi X(1)}\right] = \Phi(\xi)$

$$\Phi(\xi) \approx \inf .div. \quad \Rightarrow \quad \Phi^{1/n}(\xi) \quad \approx \quad characteristic \quad function$$

comp.Poisson $\left\{X_n(t); t \geq 0\right\} \approx \; \Phi_{X_n(1)}(\xi) = \exp\left\{n.\left[\Phi^{1/n}(\xi) - 1\right]\right\}$

It can e proved that:

$$\exp\{n.[\Phi^{1/n}(\xi) - 1]\} \xrightarrow{n \to \infty} \Phi(\xi)$$

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2) $\left\{X(t); t \geq 0\right\} \approx$ *Feller – Markov pr. with* $\left\{D_t(x, \Gamma); t \geq 0\right\}$

$$\Delta(t)u(x) = \int_R u(y) D_t(x, dy) = E\left\{u\left[X(t+\tau)\right] / X(\tau) = x\right\} \begin{cases} \Delta(0) = \Im \\ \Delta(s+t) = \Delta(s)\Delta(t) \\ \left\|\Delta(t)u\right\| \leq \left\|u\right\| \\ \Delta(t) \xrightarrow{t \downarrow 0} \Im \end{cases}$$

$$u(x) = I_\Gamma(x) \quad \to \quad \Delta(t) I_\Gamma(x) = D_t(x, \Gamma) = \Pr\left\{X(t+\tau) \in \Gamma / X(\tau) = x\right\};$$

pseudo-Poisson process $\left\{X_h(t); t \geq 0\right\}$ with transition operator $\Delta_h(t)$:

$$\Delta_h(t) = e^{-t/h} . \sum_{n=0}^{\infty} \frac{(t/h)^n}{n!} \Delta(nh) .$$

It can be proved that:

$$\Delta_h(t) = e^{-t/h} . \sum_{n=0}^{\infty} \frac{(t/h)^n}{n!} \Delta(nh) = \exp\left\{\frac{t}{h}\left[\Delta_h - \Im\right]\right\} \xrightarrow{h \to 0^+} \Delta(t)$$

Some inferential aspects

Compound-Poisson processes

X(t) ~ compound Poisson pr. if $\Rightarrow$ $X(t)=\sum_{j=1}^{N(t)} Y_j$ $\begin{cases} N(t) \approx Poisson(\lambda t) \\ f_Y \approx \Gamma(1,\beta) \end{cases}$

$$f_{X(t)}(x)=\sum_{n=0}^{\infty} P\left[N(t)=n\right].f_{X(t)/N(t)=n}(x)=\sum_{n=0}^{\infty}\left(\frac{e^{-\alpha t}.(\alpha.t)^n}{n!}\right).f_Y^{n*}(x),$$

$$f_Y^{n*}(x)\approx\Gamma(n,\beta) \qquad \begin{cases} E\left[X(t)\right]=E\left[N(t)\right].E\left(Y_j\right)=\lambda.t/\beta \\ Var\left[X(t)\right]=2\lambda t/\beta^2 \end{cases}$$

1. Maximum likelihood inferential procedure about the parameters α and β

$$\ell(\alpha,\beta/data)\propto(\alpha.\beta)^{\sum_{k=1}^{m} n_k}.\exp\left\{-m.\alpha-\beta.\sum_{k=1}^{m}\sum_{j=1}^{n_k} y_{jk}\right\}$$

m $\sim$ number of the unitary periods of observation,

$n_k \approx$ number of arrivals in the k-th period of observation,

$\sum_{j=1}^{n_k} y_{jk} \approx$ cumulative effect in the k-th period of observation.

Maximum likelihood estimates $\begin{cases} \hat{\alpha}=\sum_{k=1}^{m} n_k / m \\ \hat{\beta}=\sum_{k=1}^{m} n_k / \sum_{k=1}^{m}\sum_{j=1}^{n_k} y_{jk} \end{cases}$

2. Bayesian inference about the parameters α and β.

Observables: S(t) = X(t) – X(t-1) = $\sum_{j=1}^{N(t)-N(t-1)} Y_{tj}$

where Y_{tj} is the j-th effect in the time interval [t-1, t] .

From a Bayesian viewpoint, when the parameters α and β are unknown:

1) the random effects Y_j and the increments of the Poisson process have to be considered *exchangeable*, that is independent conditional on every values of α and β ; moreover the two processes are to be considered mutually independent conditional on every values of α and β;

2) every information which is available prior to observe the data has to be expressed by means of a joint distribution $G(\alpha, \beta / H)$, where H denotes such an information;

3) the pooling of the prior and the data information, which we will denote by K, has to take place through the application of the Bayes theorem:

$$dG(\alpha, \beta / H \wedge K) \propto \ell(\alpha, \beta / K) \;\; dG(\alpha, \beta / H) \; .$$

We will consider two different situations:

a) the statistician has no reason to assume that the effects process $\{Y_j; j \geq 1\}$ and the Poisson count process $\{N(t); t \geq 0\}$ are connected in some way so that he assume α and β to be **stochastically independent**: $G(\alpha, \beta / H) = G_1(\alpha / H).G_2(\beta / H)$;

b) the statistician believes that there is some connection between the two processes so that he assume α and β to be **stochastically dependent**: $G(\alpha, \beta / H) \neq G_1(\alpha / H).G_2(\beta / H)$.

In the first case, sub a):

$$G(\alpha,\beta/H)=G_1(\alpha/H).G_2(\beta/H) \quad \begin{cases} G_1(\alpha/H)\approx\Gamma(\alpha/a,b) \\ G_2(\beta/H)\approx\Gamma(\beta/c,d) \end{cases}$$

$$g(\alpha,\beta/H)=g_1(\alpha/H).g_2(\beta/H)\propto\left[\alpha^{a-1}.e^{-b.\alpha}\right].\left[\beta^{c-1}.e^{-d.\beta}\right] .$$

Data:

m $\approx$ number of the unitary periods of observation,

$n_k \approx$ number of arrivals in the k-th period of observation $\longrightarrow$ $n=\sum_{k=1}^{m} n_k$

$y_k^* = \sum_{j=1}^{n_k} y_{jk} \approx$ cumulative effect in the k-th period of observation $\longrightarrow y^* = \sum_{k=1}^{m} y_k^*$

$$g(\alpha,\beta/H\wedge K)\propto \ \ell(\alpha,\beta/K).g(\alpha,\beta/H)\propto$$

$$\propto(\alpha.\beta)^n.\exp\left\{-m.\alpha-\beta.y^*\right\}.\left[\alpha^{a-1}.\beta^{c-1}.\exp\left\{-(b.\alpha+d.\beta)\right\}\right]=$$

$$=\left[\alpha^{a+n-1}.e^{-\alpha(b+m)}\right].\left[\beta^{c+n-1}.e^{-\beta(d+y^*)}\right]$$

Bayesian estimates: $E\left(\alpha/H\wedge K\right)=\dfrac{a+n}{b+m}$ and $E\left(\beta/H\wedge K\right)=\dfrac{c+n}{d+y^*}$.

Obviously, a full exploitation of the posterior joint distribution is given by the determination of the expectations of given integrable functions of the two parameters:

$$E\left\{f(\alpha,\beta)/H\wedge K\right\}=\iint_{R^2} f(\alpha,\beta).g(\alpha,\beta/H\wedge K)d\alpha d\beta \ .$$

In the second case, sub b): $G(\alpha,\beta/H)\neq G_1(\alpha/H).G_2(\beta/H)$

$$g(\alpha,\beta/H)=\sum_{i=1}^{k}u_i.\Gamma(\alpha/a_i,b_i).\Gamma(\beta/c_i,d_i)\;\;;\quad u_i\geq 0,\quad \sum_{i=1}^{k}u_i=1,$$

Suppose k = 2:

$$Cov(\alpha,\beta/H)=\left(u_1-u_1^2\right).\frac{a_1}{b_1}.\frac{c_1}{d_1}+\left(u_2-u_2^2\right).\frac{a_2}{b_2}.\frac{c_2}{d_2}-u_1.u_2\left(\frac{a_1}{b_1}.\frac{c_2}{d_2}+\frac{a_2}{b_2}.\frac{c_1}{d_1}\right),$$

so that the correlation between α, β may be either positive or negative.

For simplicity, we assume:

$$b_1=b_2=b\;,\;d_1=d_2=d\;,\;a_2=a_1+r\,,\;c_2=c_1+s\,;\quad r,s\in R\,,\;\Rightarrow$$

$$\Rightarrow\;Cov(\alpha,\beta/H)=u_1.u_2.\frac{r}{b}.\frac{s}{d}\;.$$

It can be proved that the correlation between the parameters implies a correlation between the effects process $\{Y_j;j\geq 1\}$ and the Poisson count process $\{N(t);t\geq 0\}$; more precisely we have

$$Cov\left(N_k,Y_{kj}/H\right)=-\frac{d^2}{c(c+s)}Cov(\alpha,\beta/H)\;.$$

The opinion which is expressed by means of the preceding assumptions corresponds to a compound-Poisson scheme constituted by a *partially-exchangeable process*, that is by the exchangeable processes $\{Y_j;j\geq 1\}$ and $\{N_k;k\geq 0\}$ which are mutually correlated. By the Bayes theorem:

$$g(\alpha,\beta/H\wedge K)\propto g(\alpha,\beta/H).\ell(\alpha,\beta/K)\propto$$

$$\propto\left[\sum_i u_i.\Gamma(\alpha;a_i,b_i).\Gamma(\beta;c_i,d_i)\right].\left[(\alpha.\beta)^n.e^{-m\alpha-y^*\beta}\right]\propto$$

$$\propto\left[\sum_i u_i.\left(\alpha^{a_i-1}.e^{-\alpha b_i}\right).\left(\beta^{c_i-1}.e^{-\beta d_i}\right)\right].\left[(\alpha.\beta)^n.e^{-m\alpha-y^*\beta}\right]\propto$$

$$\propto\sum_i u_i^{'}.\left(\alpha^{a_i+n-1}.e^{-\alpha(b_i+m)}\right).\left(\beta^{c_i+n-1}.e^{-\beta(d_i+y^*)}\right)$$

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