LESSON 9.

1. Dimension of K-algebras.

The purpose of this lesson is to prove Theorem 1.8 of Lesson 8. In reality we will not give a complete proof of it, but we will only enunciate the Cohen-Seidenberg theorems and then we will see how, from these and from the Normalization Lemma, the theorem follows.

1.1. **Prime ideals of integral extensions.** Let $R \subset T$ be rings, R subring of T. We are interested in relations between the prime ideals of R and those of T. We are principally concerned with the case where T is integral over R, but we formulate the definitions in greater generality.

We list four properties that might hold for a pair $R \subset T$.

- (LO) *Lying over.* For any prime ideal $\mathcal{P}$ in R there exists a prime ideal $\mathcal{Q}$ in T with $\mathcal{Q} \cap R = \mathcal{P}$.
- (GU) *Going up.* Given prime ideals $\mathcal{P} \subset \mathcal{P}_0$ in R and $\mathcal{Q}$ in T with $\mathcal{Q} \cap R = \mathcal{P}$, there exists $\mathcal{Q}_0$ in T satisfying $\mathcal{Q} \subset \mathcal{Q}_0$ and $\mathcal{Q}_0 \cap R = \mathcal{P}_0$.
- (GD) *Going down.* The same with $\subset$ replaced by $\supset$.
- (INC) *Incomparable.* Two different prime ideals in T with the same contraction in R cannot be comparable.

Next Theorem 1.3 states conditions on the pair of rings that ensure the validity of the above properties. We first need some definitions.

Proposition 1.1. *Let $R \subset T$. The set $\overline{R}$ of all elements of T that are integral over R is a subring of T.*

Proof. It relies on Theorem 1.1 of Lesson 5. If $x, y \in \overline{R}$, $R[x, y]$ is a finite R-module. Therefore $x + y, x - y, xy$ are integral over R, because they all belong to $R[x, y]$. □

Definition 1.2. $\overline{R}$ is called *the integral closure of R in T*. R is called *integrally closed* in T if $\overline{R} = R$. An integral domain that is integrally closed in its field of fractions is called *normal*.

Theorem 1.3. *Let $R \subset T$ be rings with T integral over R. Then:*

1. *the pair $R \subset T$ satisfies LO, INC and GU;*
2. *if moreover R and T are integral domains and R is normal, then also GD is satisfied.*

Proof. For a proof, see for instance [Atiyah-MacDonald] or [C. Peskine, An algebraic introduction to Complex Projective Geometry, Cambridge University Press]. □

1.2. **Length of chains of prime ideals in K-algebras.** Next Theorem 1.5 is the key to prove Theorem 18.8 of Lesson 8. First we need to state some more properties of integral extensions.

Proposition 1.4. *Let $R \subset T$ be integral domains, T integral over R. Then T is a field if and only if R is a field.*

Proof. Suppose R is a field, let $y \in T, y \neq 0$. Let

$$y^n + r_1 y^{n-1} + \cdots + r_n = 0, \; r_i \in R$$

be an equation of integral dependence for y of smallest possible degree. Since T is an integral domain we have $r_n \neq 0$, so $y^{-1} = -r_n^{-1}(y^{n-1} + r_1 y^{n-2} + \cdots + r_{n-1}) \in T$. Hence T is a field.

Conversely, suppose that T is a field; let $x \in R, \; x \neq 0$. Then $x^{-1} \in T$, so it is integral over R, so that we have an equation

$$x^{-m} + s_1 x^{-m+1} + \cdots + s_m = 0, \; s_i \in R.$$

It follows that $x^{-1} = -(s_1 + s_2 x + \cdots + s_m x^{m-1}) \in R$, therefore R is a field. □

Theorem 1.5. *Let K be a field, let A be a finitely generated K-algebra, integral extension of $K[z_1, \dots, z_n]$, with $z_1, \dots, z_n$ algebraically independent over K. Then:*

a) *Every chain of prime ideals of A: $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_l$ has length $l \leq n$;*

b) *Assume that the chain is non-extendable, then $l = n$ if and only if*

$$\mathcal{P}_0 \cap K[z_1, \dots, z_n] = (0).$$

Proof. By induction on n.

If $n = 0$, then we claim that every prime ideal of A is maximal; indeed, first observe that also $A/\mathcal{P}$ is integral extension of K, because, if $a \in A$, from an equation of algebraic dependence for a over K, passing to the quotient we get a similar equation for $[a]$ over K. So by Proposition 1.4 it follows that $A/\mathcal{P}$ is a field, and we conclude that $\mathcal{P}$ is maximal. So $l = 0$.

Let $n \geq 1$, and let $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_l$ be a chain of prime ideals in A. Let $\mathcal{Q}_i = \mathcal{P}_i \cap K[z_1, \dots, z_n]$. Then, by Theorem 1.3, INC, $\mathcal{Q}_0 \subset \cdots \subset \mathcal{Q}_l$ is a chain of prime ideals in $K[z_1, \dots, z_n]$. If $l = 0$ we are done, so assume $l \geq 1$. Then $\mathcal{Q}_1$ contains a non-zero element, and, since $\mathcal{Q}_1$ is prime and $K[z_1, \dots, z_n]$ is a UFD, there exists $f \in \mathcal{Q}_1$ irreducible. So we have a chain of length $l - 1$ in $K[z_1, \dots, z_n]/(f)$, which is an integral domain:

$$\mathcal{Q}_1/(f) \subset \cdots \subset \mathcal{Q}_l/(f).$$

By the Normalization Lemma, $K[z_1, \dots, z_n]/(f)$ is an integral extension of a polynomial ring $K[y_1, \dots, y_{n-1}$. Hence, by the induction hypothesis, we have $l-1 \leq n-1$, i.e. $l \leq n$. This proves part a).

Assume now that the chain $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_l$ is not extendable. Assume $\mathcal{Q}_0 = \mathcal{P}_0 \cap K[z_1, \dots, z_n] = (0)$. Let $A' = A/\mathcal{P}_0$, $\mathcal{P}'_i = \mathcal{P}_i/\mathcal{P}_0$ for any i. The composite map $K[z_1, \dots, z_n] \hookrightarrow A \to A/\mathcal{P}_0$ is injective because $\mathcal{Q}_0 = (0)$, so $A/\mathcal{P}_0$ is integral over $K[z_1, \dots, z_n]$. We have that $K[z_1, \dots, z_n]$ is a normal ring (see the Remark at the end of the proof). Hence, we can apply Theorem 1.3 GD to this extension of rings, as follows. We have $\mathcal{Q}_0 \subsetneq \mathcal{Q}_1$. As before there exists $f \in \mathcal{Q}_1$ irreducible, generating a prime ideal with $\mathcal{Q}_0 \subsetneq (f) \subset \mathcal{Q}_1$. We have also $\mathcal{Q}_1 = \mathcal{P}'_1 \cap K[z_1, \dots, z_n]$, so by GD property there exists a prime ideal $\mathcal{N} \subset \mathcal{P}'_1$ of A' such that $\mathcal{N} \cap K[z_1, \dots, z_n] = (f)$. But the chain $\mathcal{P}'_0 \subset \mathcal{P}'_1$ is not extendable and $\mathcal{P}'_0 = (0)$, hence $\mathcal{N} = \mathcal{P}'_1$, and $(f) = \mathcal{Q}_1$. It follows that $K[z_1, \dots, z_n]/(f)$ is a subring of $A/\mathcal{P}_1$ and this is an integral extension. Again by Normalization Lemma, $K[z_1, \dots, z_n]/(f)$ is integral over a polynomial ring $K[y_1, \dots, y_{n-1}]$. Since $(0) = \mathcal{P}_1/\mathcal{P}_1 \subset \cdots \subset \mathcal{P}_l/\mathcal{P}_1$ is a non-extendable chain of prime ideals of $A/\mathcal{P}_1$, such that $(0) \cap K[y_1, \dots, y_{n-1}] = (0)$, by inductive assumption we have $l-1 = n-1$.

If $\mathcal{Q}_0 \neq 0$, let $g \in \mathcal{Q}_0$ non 0. The ring $K[z_1, \dots, z_n]/(g)$ is integral over a polynomial ring in $n-1$ variables, so the chain $\mathcal{Q}_0/(g) \subset \cdots \subset \mathcal{Q}_l/(g)$ has length at most $n-1$ and $l < n$.

□

Remark. If A is a UFD, then it is a normal ring. In particular, any polynomial ring with coefficients in a field is normal. Indeed, let $f/g \in Q(A)$ be an element of the quotient field of A, with f, g coprime. We have an equation of integral dependence

$$(f/g)^r + a_1(f/g)^{r-1} + \cdots + a_r = 0,$$

with coefficients $a_1, \dots, a_r \in A$. Multiplying everything by g^r we get:

$$f^r = -a_1 f^{r-1} g - \cdots - a_r g^r = g(-a_1 f^{r-1} - \cdots - a_r g^{r-1},$$

therefore $g | f^r$. So each irreducible factor of g divides f. Since f, g are coprime, we conclude that $g = \pm 1$ and $f/g \in A$.

1.3. **Consequences.** The following series of Corollaries of Theorem 1.5 proves the desired results and more.

Corollary 1.6. *Let A be an integral domain finitely generated as K-algebra. Let $n = tr.d.Q(A)/K$. Then*

1. **all** *non-extendable chains of prime ideals of A have length n.*
2. *The Krull dimension of A is finite and equal to n.*

(3) *Let* $\mathcal{Q} \subset \mathcal{P}$ *be two prime ideals of* A*. If*

$$\mathcal{Q} = \mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_l = \mathcal{P}$$

is a non-extendable chain of prime ideals between $\mathcal{Q}$ *and* $\mathcal{P}$*, then* $l = tr.d.Q(A/\mathcal{Q})/K - tr.d.Q(A/\mathcal{P})/K$*.*

(4) *Every maximal ideal of* A *has height* n*.*

Proof. By the Normalization Lemma there exist n algebraically independent elements $z_1, \ldots, z_n \in A$, such that A is integral over $K[z_1, \ldots, z_n]$. Since A is a domain, for any non-extendable chain of prime ideals $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_l$, we have $\mathcal{P}_0 = (0)$, hence $\mathcal{Q}_0 = \mathcal{P}_0 \cap K[z_1, \ldots . z_n] = (0)$. The proof of (1) follows by Theorem 1.5. (2) follows from (1).

To prove (3), note that, by (1), we can extend $\mathcal{Q} = \mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_l = \mathcal{P}$ to a non-extendable chain of prime ideals of A of length n:

$$(0) \subset \cdots \subset \mathcal{Q} = \mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_l = \mathcal{P} \subset \mathcal{P}_{l+1} \subset \ldots .$$

The part of the chain from $\mathcal{Q}$ up has length equal to $\dim A/\mathcal{Q} = tr.d.Q(A/\mathcal{Q})/K$, because there is a natural bijection between the set of prime ideals of $A/\mathcal{Q}$ and that of prime ideals of A containing $\mathcal{Q}$. Similarly the part from $\mathcal{P}$ up has length equal to $\dim A/\mathcal{P} = tr.d.Q(A/\mathcal{P})/K$. So (3) follows.

(4) follows because the last ideal in a non-extendable chain of prime ideals of A must be a maximal ideal. □

Corollary 1.7. *Let* $\mathcal{P} \subset K[x_1, \ldots, x_n]$ *be a prime ideal of the polynomial ring in* n *variables. Then* $\dim A/\mathcal{P} = n - ht(\mathcal{P})$*.*

Proof. Let

$$(0) = \mathcal{P}_0 \subset \cdots \subset \mathcal{P} \subset \cdots \subset \mathcal{P}_n \tag{1}$$

be a non-extendable chain of length n of prime ideals in $K[x_1, \ldots, x_n]$ passing through $\mathcal{P}$. The subchain $(0) = \mathcal{P}_0 \subset \cdots \subset \mathcal{P}$ is a non-extendable chain of prime ideals contained in $\mathcal{P}$, so it has length $ht\mathcal{P}$, whereas the subchain $\mathcal{P} \subset \cdots \subset \mathcal{P}_n$ has length $\dim A/\mathcal{P}$, so the thesis follows. □

If A is any integral domain, the property that all non-extendable chains of prime ideals of A have the same length does not hold in general. There are even examples (not easy to construct) of noetherian domains whose Krull dimension is not finite or where there are non-extendable chains of prime ideals of different lengths. The rings where the property in Corollary 1.6 (3) holds are called *catenary* rings.