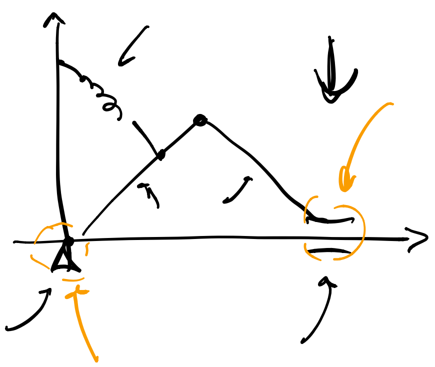
MECCANICA RAZIONALE

– LEZIONE DEL 8 APRILE 2020

– PRIMA PARTE

STATICA



compatibilmente con i vincoli

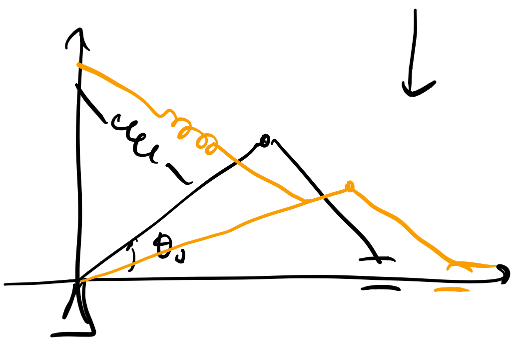
PLV , ECS

δq_i

$LV = 0$

$\downarrow$

bilanciare le forze



dinamica; partendo da θ_0, qual è

$$\boxed{\theta = \theta(t)}$$

$\Gamma = \Gamma_1$

D'Alembert : $\underbrace{\underline{F}_B = \frac{d}{dt}\underline{P}_B}_{\text{dinamica}} \iff \underbrace{\underline{F}_B - \frac{d}{dt}\underline{P}_B = \underline{0}}_{\text{statica}}$

dinamica

$(\underline{F} = m\underline{a})$

$x = x(t)$

P.L.V. $\longrightarrow$ equazioni di Lagrange

non conservativo

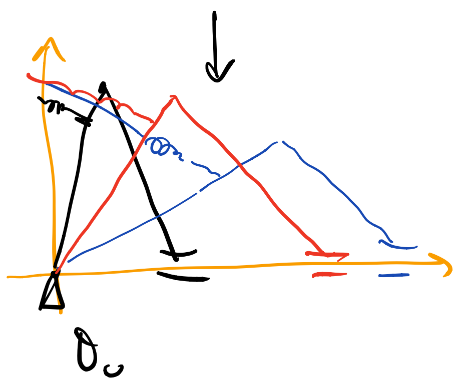
$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_i} - \frac{\partial K}{\partial q_i} = Q_i \qquad \forall i = 1, \dots, \ell$$

$\Uparrow$ forze generalizzate

K energia cinetica

$K, Q_i \longrightarrow$ eq diff 2^{nd} ordine in q_i

risolvendole, $q_i = q_i(t) \quad \forall i$



$\theta = \theta(t)$

$\theta \sim \sin \omega t + \dots$

$\theta \sim e^{-\omega_1 t} \cos \omega_2 t + \dots$

Caso conservativo

funzione lagrangiana:

$$\mathcal{L} = K - V$$

$$\boxed{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0} \qquad \forall i = 1, \dots, \ell$$

ECS $\rightsquigarrow$ E.C.D.

$$\left.\begin{array}{l}\underline{R}^e = \underline{0} \\ \underline{M}^e(O) = \underline{0}\end{array}\right\} \Rightarrow \left\{\boxed{\begin{array}{l}\underline{R}^e = \dfrac{d}{dt}\underline{P} \\ \underline{M}^e(O) = \dfrac{d}{dt}\underline{L}(O) + \underline{v}_O \wedge \underline{P}\end{array}}\right.$$

$$\underline{\underline{P}} = \sum_{B \in S} m_B \underline{v}_B = M \underline{v}_G$$ quantità di moto

$$\underline{x}_G = \frac{\sum m_B \underline{x}_B}{M} \to \underline{v}_G = \frac{\sum m_B \underline{v}_B}{M}$$

$$\underline{L}(O) = \sum_{B \in S} (\underline{x}_B - \underline{x}_O) \wedge m_B \underline{v}_B$$

momento angolare

Per $O = G$ $\qquad \underline{M}^e(G) = \dfrac{d}{dt}\underline{L}(G)$

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– SECONDA PARTE

Corpo rigido : energia cinetica e il momento angolare

Energia cinetica : $K = \frac{1}{2} \sum_{B \in S}' m_B \|\underline{v}_B\|^2$

$= \frac{1}{2} \sum_{B \in S}' m_B \, \underline{v}_B \cdot \underline{v}_B$

formula di Poisson

$\underline{v}_B = \underline{v}_0 + \underline{\omega} \wedge (\underline{x}_B - \underline{x}_0)$

$= \frac{1}{2} \sum_{B \in S} m_B \left(\underline{v}_0 + \underline{\omega} \wedge (\underline{x}_B - \underline{x}_0) \right) \cdot \left(\underline{v}_0 + \underline{\omega} \wedge (\underline{x}_B - \underline{x}_0) \right)$

$= I_1 + I_2 + I_3$

$I_1 = \frac{1}{2} \sum_{B \in R}' m_B \|\underline{v}_0\|^2$

$I_2 = \frac{1}{2} \sum_{B \in R} m_B \left(\underline{\omega} \wedge (\underline{x}_B - \underline{x}_0) \right) \cdot \left(\underline{\omega} \wedge (\underline{x}_B - \underline{x}_0) \right)$

$I_3 = \sum_{B \in R}' m_B \, \underline{v}_0 \cdot \underline{\omega} \wedge (\underline{x}_B - \underline{x}_0)$

Consideriamo i tre termini :

- $I_1 = \frac{1}{2} \sum_{B \in R} m_B \, \underline{v}_0 \cdot \underline{v}_0 = \frac{1}{2} \underline{v}_0^2 M$

Termine dovuto alla traslazione

• $I_2 = \frac{1}{2} \sum_{B \in R} m_B \left(\underbrace{\underline{\omega}}_{a} \wedge \underbrace{(\underline{x}_B - \underline{x}_0)}_{b} \right) \cdot \underbrace{\left(\underline{\omega} \wedge (\underline{x}_B - \underline{x}_0) \right)}_{c}$

$$\underline{a} \wedge \underline{b} \cdot \underline{c} = \underline{a} \cdot \underline{b} \wedge \underline{c}$$

$$= \frac{1}{2}\, \underline{\omega} \cdot \sum_{B \in R}' m_B (\underline{x}_B - \underline{x}_0) \wedge \left[\underline{\omega} \wedge (\underline{x}_B - \underline{x}_0) \right]$$

$$= \frac{1}{2}\, \underline{\omega} \cdot I_0(\underline{\omega})$$

Termine dovuto alla precessione

• $I_3 = \underline{v}_0 \cdot \underline{\omega} \wedge \sum_{B \in S}' m_B (\underline{x}_B - \underline{x}_0) =$

$$= \underline{v}_0 \cdot \underline{\omega} \wedge M (\underline{x}_G - \underline{x}_0)$$

Termine misto traslazione-precessione

$$\left[\underline{x}_G = \frac{\sum m_B \underline{x}_B}{M} \right]$$

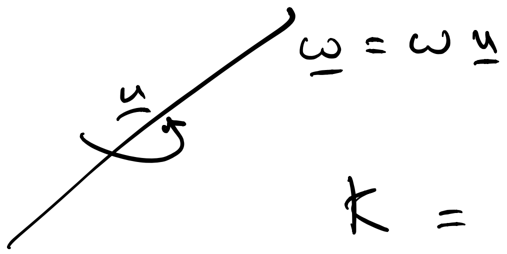
$$K = \underbrace{\frac{1}{2} M \underline{v}_0^2}_{I_1} + \underbrace{\frac{1}{2} \underline{\omega} \cdot I_0(\underline{\omega})}_{I_2} + \underbrace{\underline{v}_0 \cdot \underline{\omega} \wedge M (\underline{x}_G - \underline{x}_0)}_{I_3}$$

• Se durante il moto, il corpo rigido ha un punto fisso $\longrightarrow$ poniamo $O =$ punto fisso

$\Rightarrow \quad \underline{v}_O = \underline{0} \quad \longrightarrow \quad I_1 = I_3 = 0$

- Se il rigido ruota intorno ad un'asse fisso: scegliamo O in questo asse

$\underline{u}$ versore di questo asse $\quad \underline{\omega} = \omega\, \underline{u}$



$$K \underset{\uparrow}{=} \frac{1}{2}\, \underline{\omega} \cdot I_O(\underline{\omega}) =$$

$O \in$ asse rotazione

$$= \frac{1}{2}\, \omega\, \underline{u} \cdot I_O(\omega\, \underline{u}) \underset{\uparrow}{=} \frac{1}{2}\, \omega^2\, \underline{u} \cdot I_O \cdot \underline{u}$$

linearità

$$= \frac{1}{2}\, \omega^2\, I_z$$

$\uparrow$ momento di inerzia rispetto all'asse di rotazione z

- Se il rigido trasla: $\underline{\omega} = \underline{0}$

$\Rightarrow \quad I_2, I_3 = 0$

Riassumendo

$$K = \frac{1}{2} M v_O^2 + \frac{1}{2}\, \underline{\omega} \cdot I_O(\underline{\omega}) + \underline{v}_O \cdot \underline{\omega} \wedge M(\underline{x}_G - \underline{x}_O)$$

obbiamo i seguenti casi

- moto rigido generico :

$$K = \frac{1}{2} M v_G^2 + \frac{1}{2} \underline{\omega} \cdot I_G(\underline{\omega})$$

- moto rigido con pto fisso O (precessione)

$$K = \frac{1}{2} \underline{\omega} \cdot I_O(\underline{\omega})$$

- moto rigido con asse fisso z

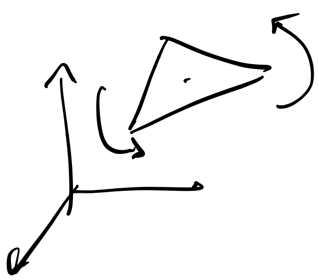
$$K = \frac{1}{2} I_z \omega^2$$

- moto rigido di traslazione

$$K = \frac{1}{2} M v_G^2$$

Caso particolare : sistemi rigidi piani

→ $\underline{\omega}$ diretto lungo la normale al piano : $\underline{e}_3$



$$\underline{\omega} = \omega \underline{e}_3$$

$$\Rightarrow \underline{\omega} \cdot I_O(\underline{\omega}) = \omega^2 \underline{e}_3 \cdot I_O(\underline{e}_3)$$

$$= \omega^2 I_{3,O}$$

Quindi per rigidi piani :

- moto rigido piano generico

$$K = \frac{1}{2} M v_G^2 + \frac{1}{2} I_{3,G} \omega^2$$

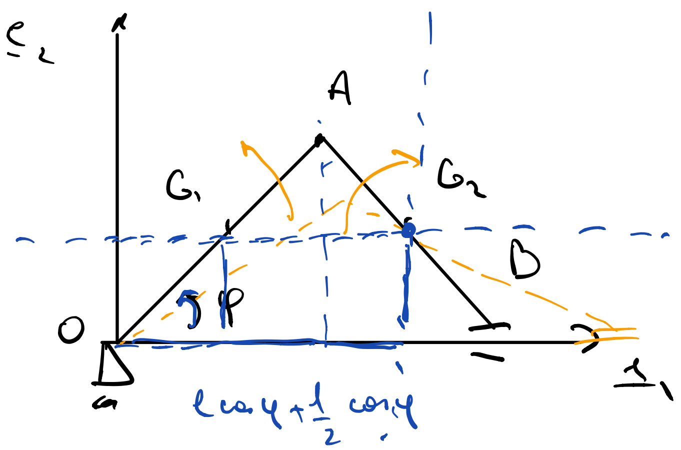
- moto rigido piano con punto O fisso

$$K = \frac{1}{2} I_{3,O} \omega^2$$

- moto rigido piano di traslazione

$$K = \frac{1}{2} M v_G^2$$

Esempio



2 aste omogenee uguali, di lunghezza l e massa m

$$K = K_{OA} + K_{AB}$$

$$\left[\text{in } K \text{ c'è } \sum_{B \in S} \right]$$

- K_{OA} moto intorno ad un punto fisso O, con $\underline{\omega} = \dot{\varphi}\, \underline{e}_3$

$$K_{OA} = \frac{1}{2} I_{3,O} \dot{\varphi}^2 = \frac{1}{2} \frac{m l^2}{3} \dot{\varphi}^2$$

- K_{AB} ~~AB~~ moto rototraslatorio con velocità angolare $-\dot{\varphi}\, \underline{e}_3$

$$K_{AB} = \frac{1}{2} m v_{G_2}^2 + \frac{1}{2} \underbrace{\frac{m}{12} l^2}_{I_{G,3}} \dot\varphi^2$$

Calcoliamo $\underline{v}_{G_2}$

$$\underline{x}_{G_2}(t) = \frac{3}{2} l \cos\varphi(t)\, \underline{e}_1 + \frac{l}{2} \sin\varphi(t)\, \underline{e}_2$$

$$\underline{v}_{G_2}(t) = \frac{d}{dt} \underline{x}_{G_2}(t) =$$

$$= \frac{l}{2} \left(-3 \sin\varphi\, \underline{e}_1 + \cos\varphi\, \underline{e}_2 \right) \frac{d\varphi}{dt}$$

$$\underline{v}_{G_2}^2 = \frac{l^2}{4} \dot\varphi^2 \left(9 \sin^2\varphi + \cos^2\varphi \right)$$

$\cos^2 + \sin^2 = 1$

$$= \frac{l^2}{4} \dot\varphi^2 \left(8 \sin^2\varphi + 1 \right)$$

$$K = \underbrace{\frac{1}{2} \frac{m l^2}{3} \dot\varphi^2}_{K_{OA}} + \underbrace{\overbrace{\frac{1}{2} m \frac{l^2}{4} \dot\varphi^2 \left(8 \sin^2\varphi + 1 \right)}^{\frac{1}{2} m v_c^2} + \frac{1}{2} \frac{m l^2}{12} \dot\varphi^2}_{K_{AB}}$$

$$= \frac{1}{2} m l^2 \dot\varphi^2 \left(2 \sin^2\varphi + \frac{2}{3} \right)$$

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– TERZA PARTE

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energia cinetica $K \longrightarrow \mathcal{L} = K - V$

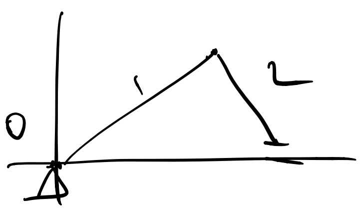
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \qquad \forall\, i = 1, \dots, \ell$$

→ calcolare K & V

Rigidi piani $\quad K = \frac{1}{2} M v_G^2 + \frac{1}{2} I_{3,G}\, \omega^2$

$$K = \frac{1}{2} I_{3O}\, \omega^2$$

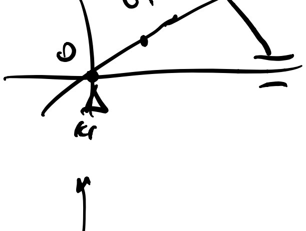
$$K = \frac{1}{2} M v_G^2$$



$K = K_1 + K_2$

K_1 ↓ rigido con O fisso

K_2 ↓ rigido moto generico



$$I_{3,O} = I_{3,G_1} + M\, d_{OG_1}^2$$

Momento angolare $\underline{L}(O)$: dipende da O

1) Supponiamo $O \in R$. Allora, usiamo la formula di Poisson $\underline{v}_B = \underline{v}_0 + \underline{\omega} \wedge (\underline{x}_B - \underline{x}_0)$

$$\underline{L}(O) = \sum_{B \in R}' (\underline{x}_B - \underline{x}_0) \wedge m_B \underline{v}_B$$

$$= \sum_{B \in R}' m_B (\underline{x}_B - \underline{x}_0) \wedge \left[\underline{v}_0 + \underline{\omega} \wedge (\underline{x}_B - \underline{x}_0) \right]$$

$$= \left[\sum_{B \in R} m_B (\underline{x}_B - \underline{x}_0) \right] \wedge \underline{v}_0 +$$

$$+ \sum_{B \in R}' m_B (\underline{x}_B - \underline{x}_0) \wedge \left[\underline{\omega} \wedge (\underline{x}_B - \underline{x}_0) \right]$$

$$= M (\underline{x}_G - \underline{x}_0) \wedge \underline{v}_0 + I_0 (\underline{\omega})$$

Termine di Traslazione

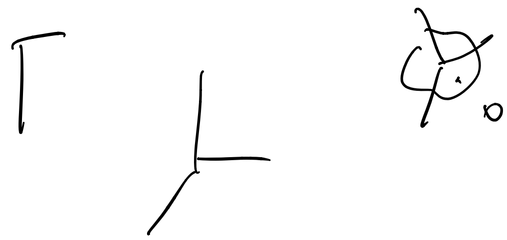
Termine di precessione

$$\underline{L}(O) = M (\underline{x}_G - \underline{x}_0) \wedge \underline{v}_0 + I_0 (\underline{\omega})$$

Casi particolari

- O fisso : $\underline{v}_O = \underline{0}$ $\quad \underline{L}(O) = I_O(\underline{\omega})$
- $O \equiv G \Rightarrow \underline{L}(G) = I_G(\underline{\omega})$

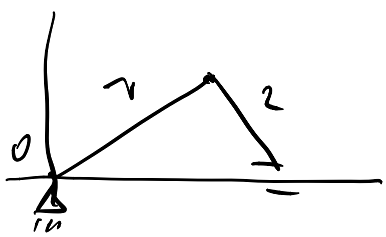
2) Se O non appartiene al rigido R non possiamo usare la formula di Poinsot



$\underline{v}_B = \underline{v}_O + \underline{\omega}\wedge(\underline{x}_B - \underline{x}_O)$

$O \notin R$

$A \in R$



$$\underline{L}(O) = \underline{L}_1(O) + \underline{L}_2(O)$$

$O \in R_1 \quad O \notin R_2$

$$\underline{L}(O) = \sum_{B\in R} (\underline{x}_B - \underline{x}_O)\wedge m_B \underline{v}_B =$$

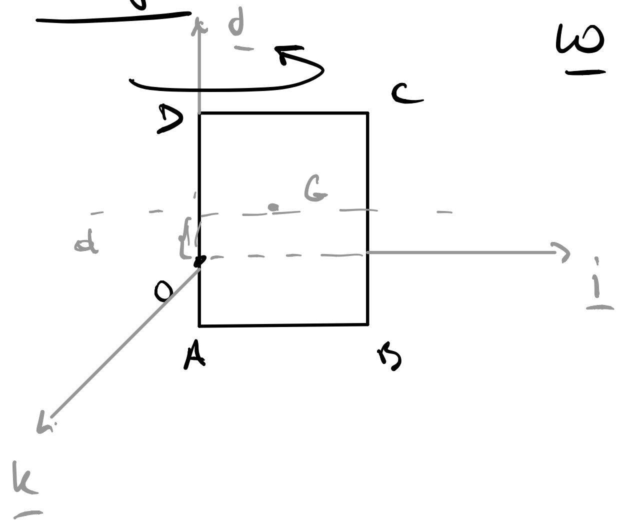
$$= \sum_{B\in R}{}' \left[(\underline{x}_B - \underline{x}_A) + (\underline{x}_A - \underline{x}_O) \right] \wedge m_B \underline{v}_B$$

$$= \underline{L}(A) + (\underline{x}_A - \underline{x}_O) \wedge \underbrace{\sum_{B\in R}{}' m_B \underline{v}_B}_{\underline{P} = M \underline{v}_G}$$

$$= \underline{L}(A) + (\underline{x}_A - \underline{x}_O) \wedge M \underline{v}_G$$

$$\Rightarrow \boxed{\underset{O\notin R}{\underline{L}(O)} = \underset{A\in R}{\underline{L}(A)} + (\underline{x}_A - \underline{x}_O)\wedge M \underline{v}_G}$$

Esempio



$\underline{\omega} = \dot{\varphi}\, \underline{j}$

rettangolo omogeneo in rotazione intorno ad AD

O : cerniera cilindrica

Calcoliamo $\underline{L}(O)$. Siccome O è fisso

$$\underline{L}(O) = I_O(\underline{\omega}) = \begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{12} & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix} \begin{pmatrix} 0 \\ \dot{\varphi} \\ 0 \end{pmatrix} =$$

$$= I_{12}\, \dot{\varphi}\, \underline{i} + I_{22}\, \dot{\varphi}\, \underline{j}$$

Se $I_{12} \neq 0$, allora $\underline{L}(O)$ non è parallelo a $\underline{\omega}$ $\left(\underline{\omega} = \dot{\varphi}\, \underline{j} \right)$

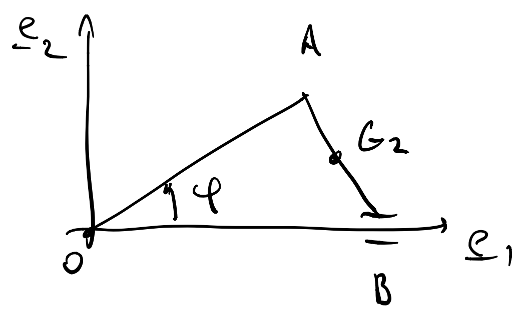
Se $I_{12} \neq 0$: $\underline{j}$ non è asse principale di inerzia relativo ad O

Complicazioni per rigidi 3D ; se andiamo a vedere rigidi piani $\longrightarrow$ $\underline{\omega}$ ortogonale al piano e l'asse ortogonale è sempre principale

$$\to \quad \underline{L}(O) = I_O(\underline{\omega}) = I_{3,O}\,\underline{\omega}$$

↑ $\underline{e}_3$ ortogonale al piano

Esempio



$$\underline{L}(O) = \underline{L}_{OA}(O) + \underline{L}_{AB}(O)$$

$O \in OA \qquad O \notin AB$

$$\underline{L}_{OA}(O) = I_{3,O}\,\dot\varphi\,\underline{e}_3 = \frac{m\ell^2}{3}\dot\varphi\,\underline{e}_3$$

$$\underline{L}_{AB}(O) = \underline{L}_{AB}(G_2) + (\underline{x}_{G_2} - \underline{x}_O)\wedge m\,\underline{v}_{G_2}$$

$O \notin AB$

$G_2 \in AB$

$G_2 \in AB$ scelta conveniente

$$\underline{L}_{AB}(G_2) = I_G(\underline{\omega}) = I_{3,G}\,\underline{\omega} = \frac{m\ell^2}{12}(-\dot\varphi)\,\underline{e}_3$$

$$\underline{\omega} = -\dot\varphi\,\underline{e}_3$$

$$(\underline{x}_{G_2} - \underline{x}_O)\wedge m\,\underline{v}_{G_2} =$$

$$= \begin{Vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \frac{3}{2}\ell\cos\varphi & \frac{\ell}{2}\sin\varphi & 0 \\ -m\frac{3}{2}\ell\sin\varphi\dot\varphi & m\frac{\ell}{2}\cos\varphi\dot\varphi & 0 \end{Vmatrix} = \frac{3}{4}m\ell^2\dot\varphi\cos^2\varphi\,\underline{e}_3 + \frac{3}{4}m\ell^2\dot\varphi\sin^2\varphi\,\underline{e}_3$$

$$= \frac{3}{4} m \ell^2 \dot{\varphi} \, \underline{e}_3$$

$$\underline{L}(0) = \frac{m\ell^2}{3} \dot{\varphi} \, \underline{e}_3 - \frac{m\ell^2}{12} \dot{\varphi} \, \underline{e}_3 + \frac{3}{4} m \ell^2 \dot{\varphi} \, \underline{e}_3$$

$$= m \ell^2 \dot{\varphi} \, \underline{e}_3$$