EXPONENTIAL DISTRIBUTION

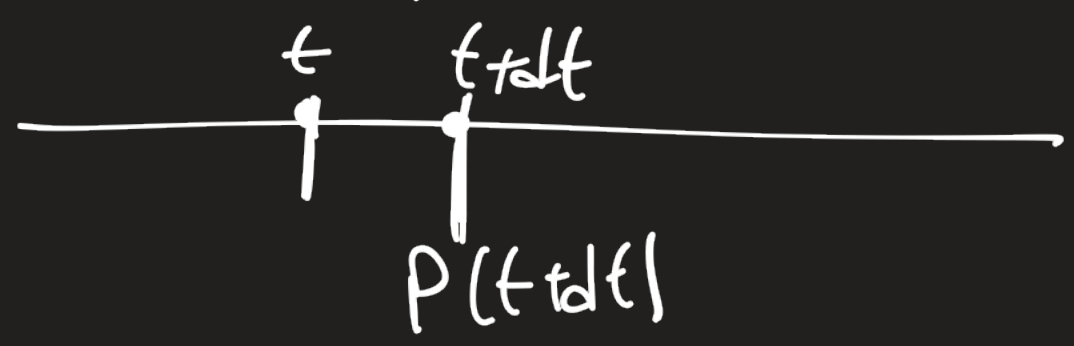
- FIRING TIME (TIME OF OCCURRENCE) OF AN EVENT

THE PROBABILITY OF AN EVENT FIRING BETWEEN TIME $[t, t+dt]$ IS EQUAL TO λdt AND INDEPENDENT OF t.

$\boxed{\lambda > 0}$ – RATE OF THE EVENT

T - R.V. denoting time of occurrence

$P(t) = P(T > t)$ SURVIVAL PROBABILITY



- $\underbrace{P(t+dt)}_{\text{event not happened } \le t+dt} = \underbrace{P(t)}_{\text{not} \le t} \cdot \underbrace{(1 - \lambda dt)}_{\text{not in } [t, t+dt]}$

- $\frac{dP(t)}{dt} = -\lambda P(t) \Rightarrow P(t) = P(0) e^{-\lambda t} = e^{-\lambda t}$

$P(T \le t) = 1 - P(T \ge t) = 1 - e^{-\lambda t}$

$$P(T\le t) = 1 - e^{-\lambda t}$$

$$p(t) = \lambda e^{-\lambda t} \quad , t \ge 0 \qquad , T \sim Exp(\lambda)$$

$$\mathbb{E}[T] = \frac{1}{\lambda} \qquad VAR[T] = \frac{1}{\lambda^2}$$

MEMORYLESS PROPERTY

$T \sim Exp(\lambda)$

$$P(T > s+t \mid T > s) = \frac{P(T > t+s)}{P(T > s)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t} = P(T > t)$$

EXP is the unique distribution with this property.

RACE CONDITION

T_k , $k \in I$ countable. $T_k \sim \mathrm{EXP}(q_k)$ INDEPENDENT s.t. $q = \sum_{k \in I} q_k < \infty$.

- $T = \inf_k T_k$, $K = \operatorname{argmin}_k T_k$.

$P(K \text{ is defined}) = 1$, T and K are independent

$T \sim \mathrm{EXP}(q)$ and $P(K=k) = \dfrac{q_k}{q}$

$K = k$ iff $T_k < T_j$, $\forall j \neq k$

$$p(K=k \text{ and } T \geq t) = p(T_k > t \text{ and } T_j > T_k , \forall j \neq k)$$

$$= \int_t^\infty q_k e^{-q_k \cdot s} p(T_j > s \text{ for } j \neq k)\, ds$$

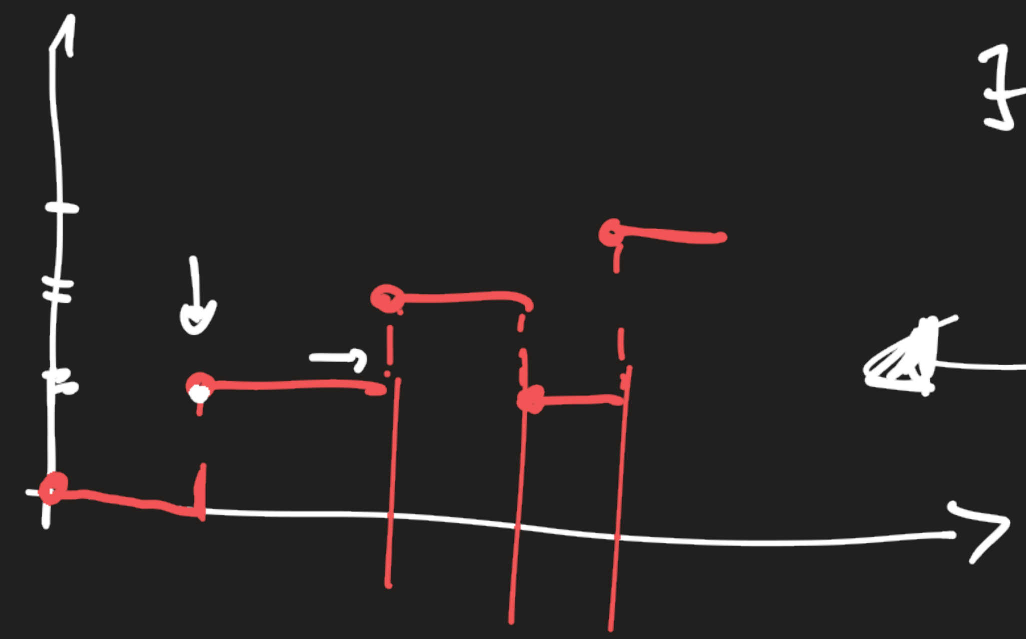
$$= \int_t^\infty q_k \underline{e^{-q_k s}} \prod_{j \neq k} e^{-q_j s}\, ds$$

$$= \frac{q_k}{q} \int_t^\infty q \cdot e^{-qs}\, ds = \frac{q_k}{q}\, e^{-qt} = \overset{q_k/q}{P(K=k)} \cdot \overset{\mathrm{EXP}(q)}{P(T \geq t)}$$

CONTINUOUS-TIME MARKOV CHAINS

S · COUNTABLE is the STATE SPACE.

$(X_t)_{t\geq 0}$, $X_t \in S$, where X_t, as a function of t, to be CADLAG.



$\exists \lim_{t\to \bar{t}^-} X_t$, $X(\bar{t}) = X(\bar{t}^+)$

X_t: CONTINUOUS TIME STOCHASTIC PROCESS.

$\wr$

X_t is determined by its FINITE DIMENSIONAL DISTRIBUTIONS

$\boxed{P(X_{t_0}, \dots, X_{t_n})}$ $\forall\, t_0<t_1<\dots<t_n \in \mathbb{R}_{\geq 0}$, $\forall n$

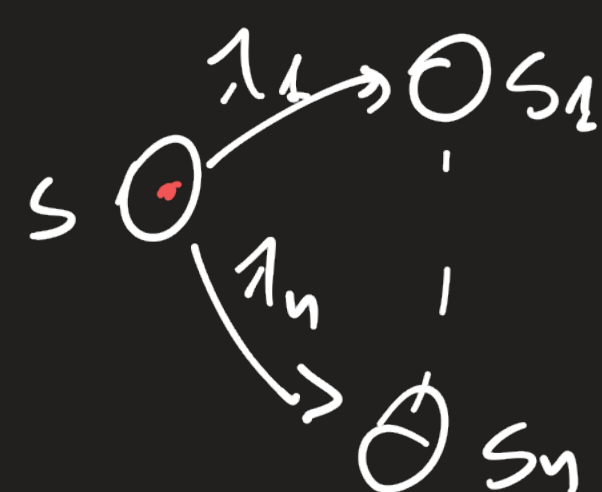
MEMORYLESS PROPERTY

$$P(X_{t_n}=s_n \mid X_{t_0}=s_0, \dots, X_{t_{n-1}}=s_{n-1}) = P(X_{t_n}=s_n \mid X_{t_{n-1}}=s_{n-1})$$

$(X_t)_{t\geq 0}$ is a CTMC iff

X_t is a CT stochastic process with cadlag sample paths & X_t has the memoryless property.

CTMC $\leftrightarrow$ LABELED GRAPHS S vertices



RACE CONDITION:

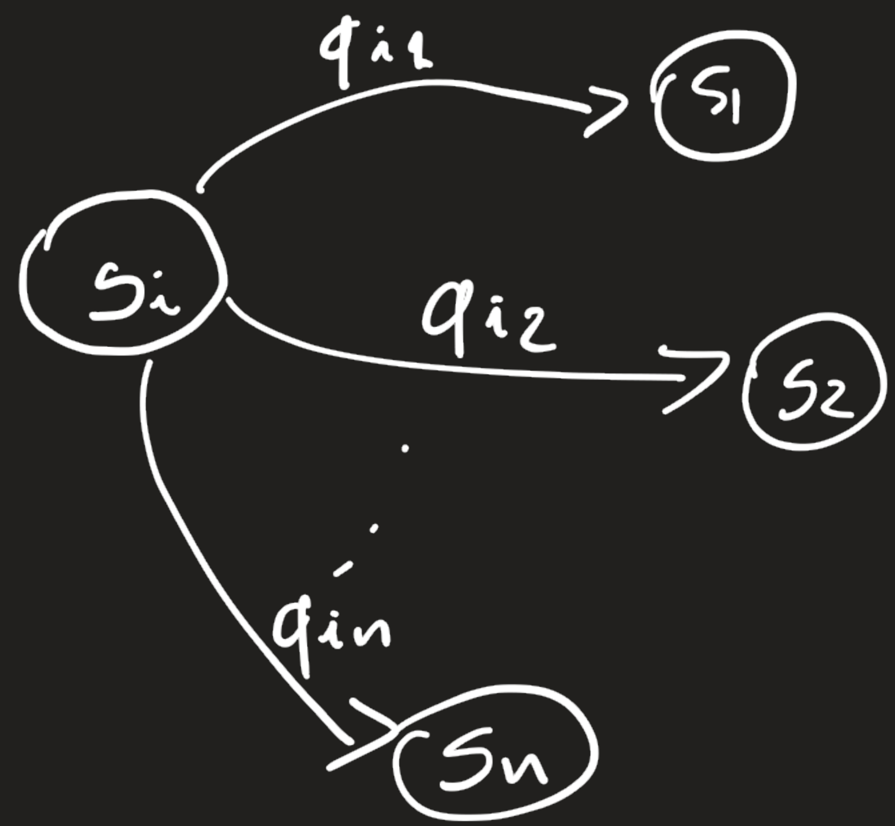
ARRANGE RATES IN A MATRIX $Q = (q_{ij})$

$q_{ij} \geq 0$, rate of jump from s_i to s_j, $\mathrm{Exp}(q_{ij})$

$q_{ii} = -\sum_{i \neq j} q_{ij}$, $-q_{ii} \geq 0$ is THE EXIT RATE

Rows of Q sum up to zero.

JUMP CHAIN & HOLDING TIMES



$q_i = \sum_{j=1}^{n} q_{ij}$ EXIT RATE OF STATE i.

Race condition:

- EXECUTE TRANSITION j WITH PROBABILITY $\frac{q_{ij}}{q_i}$
- WAIT FOR $Exp(q_i)$

THE CTMC X_t FACTORIZES IN [Y_n and T_n are independent cond. on Y_{n-1}]

- Y_n, a DTMC - THE JUMP CHAIN - with transition matrix

$$\Pi = (\pi_{ij}) \quad \pi_{ij} = \frac{q_{ij}}{q_i}, \pi_{ii} = 0$$

- A SEQUENCE OF JUMP TIMES τ_n : time of nth jump

$T_n = \tau_n - \tau_{n-1}$, n^{th} HOLDING TIME, $T_n \sim Exp(q_{Y_{n-1}})$

$\Rightarrow X_t = Y_n$ for $\tau_n \leq t < \tau_{n+1}$

KOLMOGOROV EQUATIONS

$$P(t):\ P_{ij}(t) = p(X_t = s_j \mid X_0 = s_i) \qquad t \in \mathbb{R}_{\geq 0}$$

$$P_{ij}(t+r) = p(X_{t+r} = s_j \mid X_0 = s_i) = \sum_{s_k} p(X_{t+r} = s_j, X_t = s_k \mid X_0 = s_i)$$

$$= \sum_{s_k} p(\underbrace{X_{t+r} = s_j \mid X_t = s_k, \cancel{X_0 = s_i}}_{\text{TIME HOMOGENEITY:}}) \, p(X_t = s_k \mid X_0 = s_i)$$

$$= \sum_{s_k} p(X_r = s_j \mid X_0 = s_k) \, p(X_t = s_k \mid X_0 = s_i)$$

$$= \sum_{s_k} P_{ik}(r) P_{kj}(t)$$

CHAPMAN-KOLMOGOROV EQN.

$$\Rightarrow P(t+r) = P(r) \cdot P(t) = P(t) P(r)$$

SEMIGROUP PROPERTY

$P(t) \equiv$ TRANSIENT PROBABILITY

$P(dt)$? $P_{ij}(dt) = q_{ij}\, dt$

$P_{ii}(dt) = 1 - \sum_{j \neq i} q_{ij}\, dt$

$$\boxed{P(dt) = \underline{I} + \underline{Q} \cdot dt}$$

$$P(t+dt) = P(t)P(dt) = P(t)(I + Q\,dt) = P(t) + P(t)Q\,dt$$

$$\frac{dP(t)}{dt} = P(t)\,\underline{Q} \quad \text{FORWARD KOLMOGOROV EQUATION}$$

$$P(t+dt) = P(dt)P(t) = P(t) + QP(t)\,dt$$

$$\frac{dP(t)}{dt} = QP(t) \quad \text{BACKWARD KOLMOGOROV EQUATIONS.}$$

$$P(0) = I$$

$p_0 = P(X_0)$ $p(t) = p_0 P(t)$

$$\frac{d}{dt}p(t) = \frac{d}{dt}p_0 P(t) = p_0 P(t) \cdot Q = p(t) \cdot Q$$

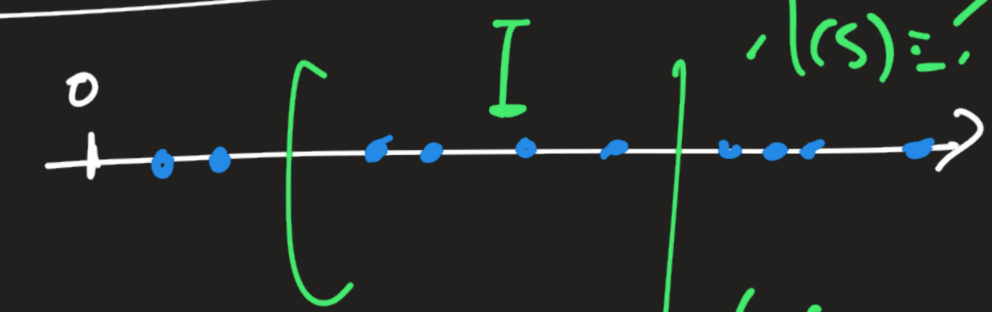
Poisson Process

$$0 \xrightarrow{\lambda} 1 \xrightarrow{\lambda} 2 \xrightarrow{\lambda} 3 \xrightarrow{\lambda}$$

$$X_t = N_\lambda(0,t) = \underbrace{\text{Poisson}(\lambda \cdot t)}$$

$$p(Y_\lambda = n) = \frac{e^{-\lambda}\lambda^n}{n!}$$

$\lambda(s) = \lambda$



$$P(N(I)) = \text{Poisson}\left(\underbrace{\int_I \lambda(s)\,ds}_{\lambda \cdot \text{length}(I)}\right)$$

Generating function

$$G(z) = \mathbb{E}\left[z^{Y_\lambda}\right] = e^{\lambda(z-1)}$$

$$G_t(z) = \mathbb{E}\left[z^{N_\lambda(0,t)}\right]$$

. Markov Property $N_\lambda(0,t+s) = N_\lambda(0,t) + N_\lambda(0,s)$

$$G_{t+dt}(z) = \underbrace{\mathbb{E}\left[z^{N(0,t)}\right]}_{G_t(z)}\underbrace{\mathbb{E}\left[z^{N(t,t+dt)}\right]}_{\left((1-\lambda dt)z^0 + \lambda dt\, z^1\right)} \Rightarrow G_{t+dt}(z) = G_t(z) + \lambda(z-1)G_t(z)\,dt$$

$$\Rightarrow \frac{dG_t(z)}{dt} = \lambda(z-1)G_t(z) \Rightarrow G_t(z) = e^{\lambda t(z-1)}, \quad N(0,0) = 0$$

STEADY STATE OF CTMC

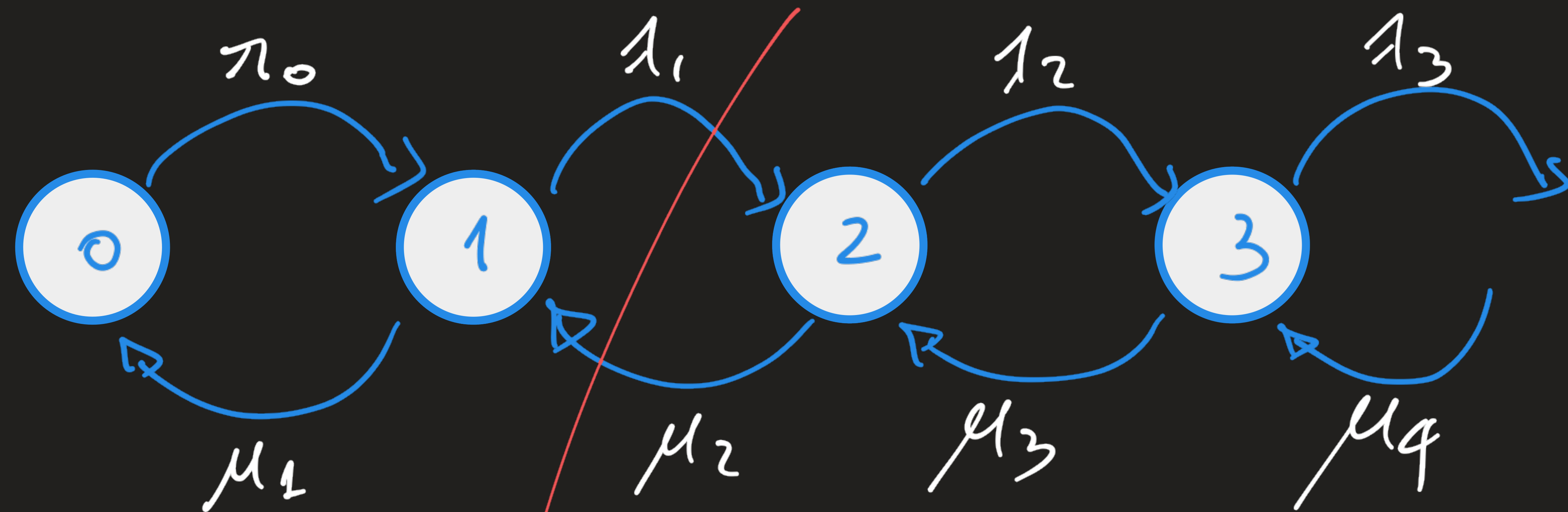
- S FINITE and let Q be given
- A measure (probability distribution on S) π is INVARIANT iff

$$\underbrace{\pi \cdot Q = 0}_{\text{equilibrium for KOLMOGOROV equations}} \qquad \left[\frac{dp}{dt} = pQ = 0 \right]$$

- IF Q is IRREDUCIBLE (THE CTMC GRAPH IS STRONGLY CONNECTED) then π is UNIQUE
- FOR IRREDUCIBLE CTMC, with Q as infin. generator, with invar. measure π, then

$$\lim_{t \to \infty} P_{ij}(t) = \pi_j$$

- NO APERIODICITY! $\rightsquigarrow$ due to exp. distributed random delays.



π = stw dist.

$$\boxed{\pi_i \cdot \lambda_i = \pi_{i+1}\,\mu_{i+1}}$$

$$\pi_K = \prod_{i=0}^{K-1} \frac{\lambda_i}{\mu_{i+1}} \pi_0 \quad \Rightarrow \quad \pi_0 = \left(1 + \sum_{K=1}^{\infty} \prod_{i=0}^{K-1} \frac{\lambda_i}{\mu_{i+1}}\right)^{-1}$$

TRANSIENT PROBABILITY

$\underline{P(t)}\ \{P_{ij}(t) = \mathbb{P}(X_t = s_j \mid X_0 = s_i)\}$

$$\frac{d}{dt}P(t) = P(t)\cdot Q$$

$$P(t) = e^{Qt} := \sum_{n=0}^{\infty} \frac{t^n Q^n}{n!}$$ MATRIX EXPONENTIAL

UNIFORMIZATION

$\underbrace{\sup_i q_i}_{\text{max exit rate}} \leq \lambda$

$\Pi = I + \frac{Q}{\lambda}$ is a stochastic matrix

- NUMBER OF FIRINGS UP TO TIME t: $N(0,t)$ IS POISSON(λt)
- Y DTMC with transition matrix Π

$\Rightarrow \boxed{X_t = Y_{N(0,t)}}$

$$P(t) = \sum_{n=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!}\Pi^n$$

TIME INHOMOGENEOUS EXPONENTIAL

$T \sim \mathrm{Exp}(\lambda)$, $\lambda = \lambda(t)$ $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$

$$\underbrace{\Lambda(t) = \int_0^t \lambda(s)\,ds}_{\text{CUMULATIVE RATE}} \qquad \left(\lambda \text{ CONSTANT} \Rightarrow \Lambda(t) = \lambda \cdot t \right)$$

$$p(T \leq t) = 1 - e^{-\Lambda(t)} \qquad p(T \geq t) = e^{-\Lambda(t)}$$

$$p(t) = \lambda(t)\, e^{-\Lambda(t)} \qquad \left[\Lambda(t) = -\log U \quad \text{INVERSION METHOD} \right]$$