Invariant Measures and Steady State

Invariant Measure

Consider a CTMC with rate matrix Q and finite state space S. An invariant measure for the CTMC is a probability distribution π satisfying

$$\pi Q = 0.$$

If Q is irreducible (has a strongly connected graph), then it has a unique invariant measure.

Steady state behaviour

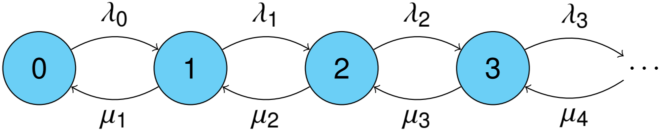
Consider an irreducible CTMC with rate matrix Q and finite state space S, and let π be its invariant probability distribution. Then, for each $s_i, s_j \in S$,

$$\lim_{t \to \infty} P_{ij}(t) = \pi_j.$$

Notice that aperiodicity is not required. Why?

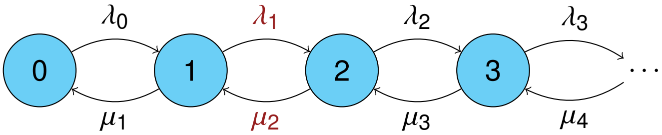
EXAMPLE: BIRTH-DEATH PROCESS

A birth-death process is a CTMC on $S = \mathbb{N}$ with birth rate λ_i (from i to $i + 1$) and death rate μ_i (from i to $i - 1$).



EXAMPLE: BIRTH-DEATH PROCESS

A birth-death process is a CTMC on $S = \mathbb{N}$ with birth rate λ_i (from i to $i + 1$) and death rate μ_i (from i to $i - 1$).



To derive the steady state π, we can use the fact that the net flow along each cut must be zero (why?):

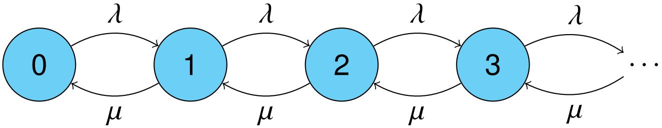
$$\pi_i \lambda_i = \pi_{i+1} \mu_{i+1}$$

Hence we get:

$$\pi_k = \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}} \pi_0; \qquad \pi_0 = \left(1 + \sum_{k=1}^{\infty} \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}} \right)^{-1}$$

EXAMPLE: BIRTH-DEATH PROCESS

Consider a birth-death process with constant birth rate λ and constant death rate μ. It is the model of an M/M/∞ queue.



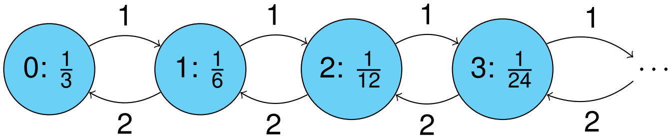
$$\pi_k = \left(\frac{\lambda}{\mu}\right)^k \pi_0; \qquad \pi_0 = \left(1 + \sum_{k=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^k\right)^{-1}$$

- If $\lambda \geq \mu$, then $\pi_0 = 0 = \pi_k$. No state is positive recurrent, there is no invariant measure. The chain escapes to infinity.
- If $\lambda < \mu$, then $\pi_0 = \frac{1-\lambda/\mu}{2-\lambda/\mu}$ and $\pi_k = \left(\frac{\lambda}{\mu}\right)^k \frac{1-\lambda/\mu}{2-\lambda/\mu}$

Example: birth-death process

If $\lambda < \mu$, then $\pi_0 = \frac{1-\lambda/\mu}{2-\lambda/\mu}$ and $\pi_k = \left(\frac{\lambda}{\mu}\right)^k \frac{1-\lambda/\mu}{2-\lambda/\mu}$

Assume $\lambda = 1$, $\mu = 2$.



MATRIX EXPONENTIAL

The solution of the forward Kolmogorov equation $\frac{dP(t)}{dt} = P(t)Q$, for a generic CTMC, can be given in terms of the matrix exponential

$$P(t) = e^{Qt} = \sum_{n=0}^{\infty} \frac{t^n Q^n}{n!}.$$

However, numerical computation of the series expansion is numerically unstable.

UNIFORMIZATION

A more efficient strategy is to solve the uniformized CTMC.
Let $\lambda \geq \max_i\{-q_{ii}\}$.
Then one considers a CTMC with jump chain $Y(n)$ with matrix

$$\Pi = I + \frac{1}{\lambda}Q,$$

and uniform exit rate λ.
The number of fires of this CTMC up to time t is a Poisson process $N_\lambda(0,t)$, and so

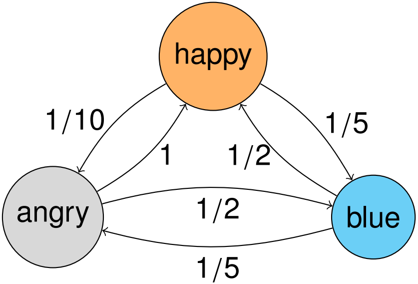
$$X(t) = Y_{N(0,t)} = Y_{\mathcal{Y}(\lambda t)}.$$

It follows that

$$P(t) = \sum_{n=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!}\Pi^n,$$

which can be truncated above (and below) by bounding the Poisson r.v.

A SIMPLE EXAMPLE: THE MOOD CHAIN



Upper bound on exit rate: 2

$$P(t) = \sum_{n=0}^{\infty} \frac{e^{-2t}(2t)^n}{n!} \Pi^n$$

$$\Pi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -\frac{3}{10} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{2} & -\frac{7}{10} & \frac{1}{5} \\ 1 & \frac{1}{2} & -\frac{3}{2} \end{pmatrix} = \begin{pmatrix} \frac{17}{20} & \frac{2}{20} & \frac{1}{20} \\ \frac{5}{20} & \frac{13}{20} & \frac{2}{20} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Outline

TIME-INHOMOGENEOUS EXPONENTIAL

DEFINITION

A exponential random variable $T \sim Exp(\lambda)$ has time inhomogeneous rate iff $\lambda = \lambda(t)$ is a function $\lambda : [0, \infty[\to \mathbb{R}^+$.

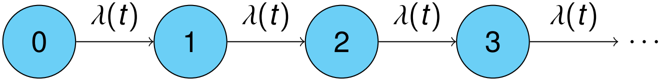
- Cumulative rate is $\Lambda(t) = \int_0^t \lambda(s) ds$
- Cdf is $\mathbb{P}(T < t) = 1 - e^{-\Lambda(t)}$
- Survival probability is $\mathbb{P}(T > t) = e^{-\Lambda(t)}$

INVERSION METHOD

One can simulate unidimensional random variables by sampling a uniform r.v. $U \in [0, 1]$, and then finding t^* such that $t^* = \inf_t \mathbb{P}(T \leq t) = U$.
For a time-inhomogeneous $Exp(\lambda(t))$, one has to solve $e^{-\Lambda(t)} = U$, iff $\Lambda(t) = -\log U = \xi$, with $\xi \sim Exp(1)$.
If λ is constant, then $\Lambda(t) = \lambda t$, and one has $t = -\frac{1}{\lambda} \log(U)$.
In general, one can either integrate $\lambda(t)$ or the equivalent ODE $\frac{d\Lambda(t)}{dt} = \lambda(t)$, and check for the root of $\Lambda(t) + \log(U)$ along the solution.

TIME-INHOMOGENEOUS POISSON PROCESS

A time-inhomogeneous Poisson process $\mathcal{N}_\lambda(0,t)$, $\lambda = \lambda(t)$, is a Poisson process with time-varying rate.



It can be shown (same generating function argument as above) that the distribution of $\mathcal{N}_\lambda(0,t)$ is *Poisson*$(\Lambda(t))$, i.e. it is the r.v.

$$\mathcal{Y}(\Lambda(t)) = \mathcal{Y}\left(\int_0^t \lambda(s)ds\right).$$

TIME-INHOMOGENEOUS CTMC

TIME-INHOMOGENEOUS CTMC

In general, if the rate matrix Q of a CTMC depends on time, $Q = Q(t)$, then the CTMC is time inhomogeneous.
The probability semigroup depends now also on the initial time: $P_{ij}(t_1, t_2) = \mathbb{P}\{X(t_2) = s_j \mid X(t_1) = s_i\}$.

FORWARD KOLMOGOROV EQUATION

$$\frac{\partial P(t_1, t_2)}{\partial t_2} = P(t_1, t_2)Q(t_2)$$

BACKWARD KOLMOGOROV EQUATION

$$\frac{\partial P(t_1, t_2)}{\partial t_1} = -Q(t_1)P(t_1, t_2)$$