Exercise 1.

Consider the theory of a real scalar field ϕ with Lagrangian

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{3!}\phi^3 \ . \tag{1}$$

1. Derive the functional generator for all connected Green's functions, $iW[J]$, up to second order in perturbative expansion in the coupling λ, i.e. $\mathscr{O}(\lambda^2)$.
2. Derive all connected Green's functions at the same order in coupling expansion, in position space, and draw the corresponding Feynman diagram.

Exercise 2.

Derive the Feynman rules for the following Lagrangian interactions:

$$\begin{aligned}
&1) \quad \mathscr{L}_I = -\frac{\lambda}{3!}\phi(x)^3 \\
&2) \quad \mathscr{L}_I = -\lambda(\phi(x)^*\phi(x))^2 \\
&3) \quad \mathscr{L}_I = C\,\phi(x)^2(\partial_\mu\phi(x))(\partial^\mu\phi(x)) \\
&4) \quad \mathscr{L}_I = ig\left[\phi(x)^*(\partial_\mu\phi(x)) - (\partial_\mu\phi(x)^*)\phi(x)\right]A^\mu(x)
\end{aligned}$$

Exercise 3.

Consider the theory of a massive spin-1 field A_μ with the Proca Lagrangian. The equation of motion is given by

$$(\Box g_{\mu\nu} - \partial_\mu\partial_\nu + m^2 g_{\mu\nu})A^\nu = 0 \ . \tag{2}$$

1. Given the propagator in the form $D_V^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} i\Pi^{\mu\nu}(k)e^{-ik(x-y)}$, find $\Pi^{\mu\nu}(k)$ such that this propagator is the inverse of the equations of motion:

$$(\Box g_{\mu\nu} - \partial_\mu\partial_\nu + m^2 g_{\mu\nu})D_V^{\nu\rho}(x-y) = ig_\mu^\rho\delta^{(4)}(x-y) \ . \tag{3}$$

2. Going to a frame where the vector's momentum is $k^\mu = (E_k, 0, 0, |\vec{k}|)$ with $E_k = \sqrt{|\vec{k}|^2 + m^2}$, the three polarization vectors are

$$\varepsilon_1^\mu(k) = (0,1,0,0) \ , \quad \varepsilon_2^\mu(k) = (0,0,1,0) \ , \quad \varepsilon_L^\mu(k) = \frac{1}{m}(|\vec{k}|, 0, 0, E_k) \ . \tag{4}$$

   Show that

$$\sum_\lambda \varepsilon_\lambda^\mu(k)\varepsilon_\lambda^\nu(k) = -g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2} \ . \tag{5}$$