MECCANICA RAZIONALE

– LEZIONE DEL 14 APRILE 2020

– PRIMA PARTE

Problema della dinamica

Statica		Dinamica
P.L.V.	Principio di D'Alembert	Equazioni di Lagrange
E.C.S.		E.C.D.

PLV $\displaystyle\sum_{B\in S} \underline{F}_B \cdot \delta \underline{x}_B = 0$ $\forall\ \delta \underline{x}_B$ virtuale invertibile

Principio: $\underline{F}_B \Rightarrow \underline{F}_B - \dfrac{d}{dt}\underline{P}_B$

$\downarrow$ manipolazioni $\left(\dfrac{d}{dt}\underline{P}_B \rightsquigarrow K\right)$

$$\frac{d}{dt}\frac{\partial K}{\partial \dot q_i} - \frac{\partial K}{\partial q_i} = Q_i$$ caso non conservativo

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$ caso conservativo

$$\mathcal{L} = K - V$$

$i = 1, \dots, l$ gradi di libertà

→ l equazioni differenziali del 2^{nd} ordine

che generalizzano $\underline{F} = m \frac{d^2 x}{dt^2}$

→ se abbiamo $x(t_0)$ e $\dot{x}(t_0)$

permettono di determinare $\underline{x} = \underline{x}(t)$

Allo stesso modo

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$ → l eq. differenziali del secondo ordine

$q_i(t_0)$, $\dot{q}_i(t_0)$ $\forall i = 1, \dots, l$ ⇒ $q_i = q_i(t)$

Allo stesso modo

E.C.S. ⟺ ECD

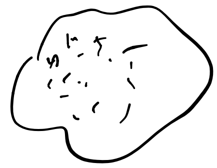
$\underline{R}^e = \underline{0}$ ⟺ $\underline{R}^e = \frac{d}{dt}\underline{P}$

$\underline{M}^e(O) = \underline{0}$ ⟺ $\underline{M}^e(O) = \frac{d}{dt}\underline{L}(O) + \underline{v}_O \wedge \underline{P}$

Problema della dinamica :

$$\mathcal{L}\ , \quad K\ , \quad \underline{P}\ , \quad \underline{L}(O)$$

$$\left(K = \frac{1}{2} \sum_{B \in S}' m_B \, \|\underline{v}_B\|^2 \right.$$



$$\left. \frac{1}{2} \int \rho(x) \, \|\underline{v}_B\|^2 \right)$$

Energia cinetica $\longrightarrow$ matrice di inerzia

Per un corpo rigido piano :

$\rightarrow$ generico $\quad K = \frac{1}{2} M v_G^2 + \frac{1}{2} I_{3,G} \, \omega^2$

$\rightarrow$ punto fisso O $\quad K = \frac{1}{2} I_{3,O} \, \omega^2$

$\rightarrow$ Traslazione $\quad K = \frac{1}{2} M v_G^2$

Momento

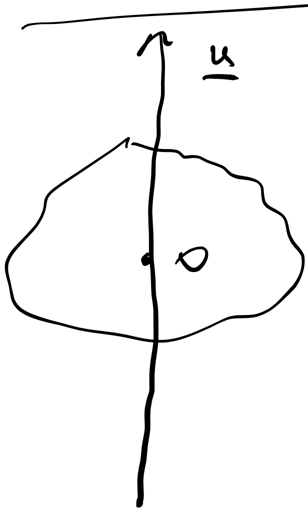
$$\underline{L}(O) = \sum_{B \in R} (\underline{x}_B - \underline{x}_O) \wedge m_B v_B$$

$$= I_O(\underline{\omega}) + M (\underline{x}_G - \underline{x}_O) \wedge \underline{v}_O$$

Termine di precessione

Termine di traslazione

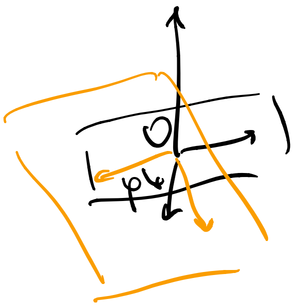
Applicazioni → corpo rigido in rotazione intorno ad un asse fisso



Consideriamo un rigido, asse di rotazione di versore $\underline{u}$, cerniera cilindrica in O

Coordinata libera φ

$\underline{e}_3 = \underline{k} = \underline{u}$



Equazioni Cardinali della Dinamica

$$\underline{R}^e = \underline{R}^{e,a} + \underline{F}_O^r = \dot{\underline{P}}$$

$\underline{R}^{e,a}$: attive

$\underline{F}_O^r$: reazioni in O

3 incognite

$$\underline{M}^e(O) = \underline{M}^{e,a}(O) + \underline{\mu}_O^r = \frac{d}{dt}\underline{L}(O)$$

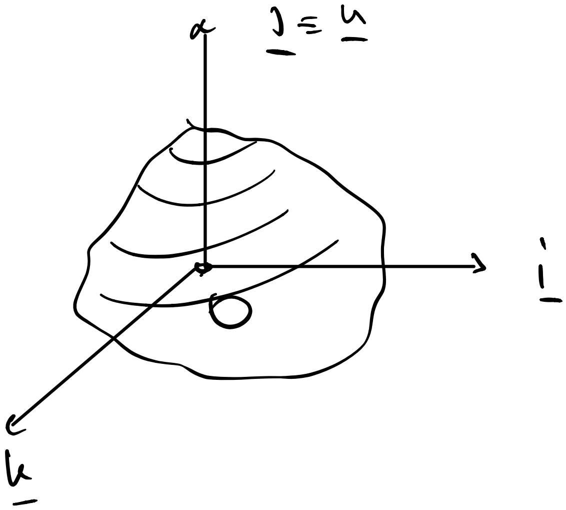
$\underline{M}^{e,a}(O)$: attive

$\underline{\mu}_O^r$: momento forze esterne di reazione: 2 incognite

[cerniera liscia: $\underline{\mu}_O^r \cdot \underline{u} = 0$ niente componente lungo $\underline{u}$]

Incognite: φ, 5 incognite di reazione

Prendiamo $S(O: \underline{i}, \underline{j}, \underline{k})$ $\quad \underline{j} \equiv \underline{u}$



$$\underline{L}(O) = I_O(\underline{\omega}) =$$

$$= I_O(\dot\varphi\, \underline{u}) =$$

$$= \dot\varphi\, I_O(\underline{u})$$

$$= \dot\varphi \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{pmatrix} \underset{\uparrow\, \underline{u} = \underline{j}}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} = \dot\varphi \left(I_{12}\, \underline{i} + I_{22}\, \underline{j} + I_{23}\, \underline{k} \right)$$

$$\underline{L}(O) = \dot\varphi \left(I_{12}\, \underline{i} + I_{22}\, \underline{j} + I_{23}\, \underline{k} \right)$$

Vogliamo $\dfrac{d}{dt} \underline{L}(O)$

Supponiamo $\underline{t}$ vettore fisso in S e dipendente dal tempo. Scriviamo $\underline{t} = \underline{x}_A - \underline{x}_B$ (prendiamo B come polo) e usiamo Poisson:

$$\frac{d}{dt} \underline{x}_A = \frac{d}{dt} \underline{x}_B + \underline{\omega} \wedge (\underline{x}_A - \underline{x}_B)$$

$$\Rightarrow \boxed{\frac{d}{dt}\underline{r} = \underline{\omega} \wedge (\underline{x}_A - \underline{x}_B) = \underline{\omega} \wedge \underline{r}}$$

$$\frac{d}{dt}\underline{L}(O) = \frac{d}{dt}\left[\dot{\varphi}\, I_O(\underline{u})\right] =$$

$$= \left(\frac{d}{dt}\dot{\varphi}\right) I_O(\underline{u}) + \dot{\varphi}\,\frac{d}{dt} I_O(\underline{u})$$

$$= \ddot{\varphi}\, I_O(\underline{u}) + \dot{\varphi}\,\underline{\omega} \wedge I_O(\underline{u})$$

$$\left[\;\omega = \dot{\varphi}\,\underline{u} = \dot{\varphi}\,\underline{j}\;\right]$$

$$= I_O(\underline{\dot{\omega}}) + \underline{\omega} \wedge I_O(\underline{\omega})$$

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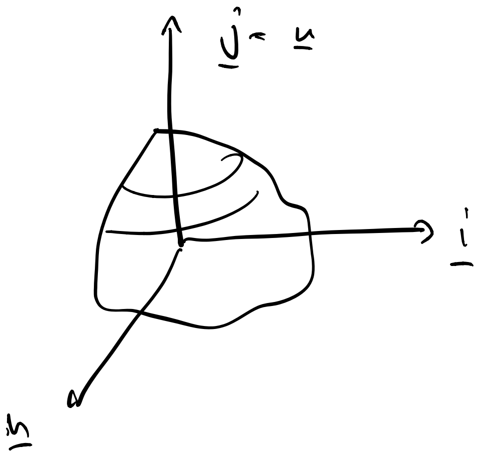
- SECONDA PARTE



$$\underline{R}^{e,a} + \underline{F}_O^{r} = \underline{\dot{P}}$$

$$\underline{M}^{e,a}_{(O)} + \underline{\mu}_O^{r} = \frac{d}{dt}\underline{L}(O)$$

$$\underline{L}(O) = I_O(\underline{\omega}) = \dot{\varphi}\, I_O(\underline{u}) \qquad \underline{\omega} = \dot{\varphi}\,\underline{u}$$

$$\rightarrow \frac{d}{dt}\underline{L}(O) = \ddot{\varphi}\, I_O(\underline{u}) + \dot{\varphi}\,\underline{\omega} \wedge I_O(\underline{u})$$

$$= I_O(\dot{\underline{\omega}}) + \underline{\omega} \wedge I_O(\underline{\omega})$$

Usiamo $I_O(\underline{u}) = I_{12}\,\underline{i} + I_{22}\,\underline{j} + I_{23}\,\underline{k}$

e sostituiamo

$$\ddot{\varphi}\, I_O(\underline{u}) + \dot{\varphi}\,\underline{\omega} \wedge I_O(\underline{u}) =$$

$$= \ddot{\varphi}\left(I_{12}\,\underline{i} + I_{22}\,\underline{j} + I_{23}\,\underline{k} \right) +$$

$$+ \dot{\varphi}^2\,\underline{u} \wedge \left(I_{12}\,\underline{i} + I_{22}\,\underline{j} + I_{23}\,\underline{k} \right)$$

$$= \ddot{\varphi}\left(I_{12}\,\underline{i} + I_{22}\,\underline{j} + I_{23}\,\underline{k} \right) +$$

$$+ \dot{\varphi}^2\left(- I_{12}\,\underline{k} + I_{23}\,\underline{i} \right)$$

$$\underline{j} \wedge \underline{i} = -\underline{k} \qquad \underline{j} \wedge \underline{k} = \underline{i}$$

$$= \left(\ddot{\varphi}\, I_{12} + \dot{\varphi}^2 I_{23} \right)\underline{i} + I_{22}\,\ddot{\varphi}\,\underline{j}$$

$$+ \left(\ddot{\varphi}\, I_{23} - \dot{\varphi}^2 I_{12} \right)\underline{k}$$

$$\underline{M}^{e,e}(O) + \underline{\mu}_o^{r} = \frac{d}{dt}\underline{L}(O)$$

$$\underline{i} \qquad \underline{M}^{e,o}(O)\cdot\underline{i} + \underline{\mu}_o^{r}\cdot\underline{i} = I_{12}\ddot{\varphi} - \dot{\varphi}^2 I_{23}$$

$$\underline{j} \qquad \underline{M}^{e,o}(O)\cdot\underline{j} = I_{22}\ddot{\varphi}$$

$$\underline{k} \qquad \underline{M}^{e,o}(O)\cdot\underline{k} + \underline{\mu}_o^{r}\cdot\underline{k} = \ddot{\varphi} I_{23} - \dot{\varphi}^2 I_{12}$$

Notiamo: se $\underline{j}$ non è asse principale di inerzia $\to$ I_{12} e I_{23} sono $\neq 0$

Momento di reazione μ_o^{r} dipende sia dalla parte statica $\underline{M}^{e,e}(O)$, e anche dalla parte di inerzia $(\ddot{\varphi}, \dot{\varphi}^2, I_{12}, I_{23})$

Momento deviatori $\to$ tendono a far deviare l'asse di rotazione $\to$ responsabili di forze di reazione della cerniera

Ad esempio: se $\underline{u}$ asse principale di inerzia

$$I_O(\underline{\omega}) = \dot{\varphi}\, I_O(\underline{u}) = \dot{\varphi}\, I_z\, \underline{u}$$

$$\rightarrow \quad \frac{d}{dt} \underline{L}(O) = \ddot{\varphi}\, I_z\, \underline{u}$$

Quindi $\quad \underline{M}^{e,a}(O) \cdot \underline{u} = I_z \ddot{\varphi}$

$(\underline{u} \equiv \underline{j})$

$$\rightarrow \quad \underline{\mu}_O^{z} = -\left(\underline{M}^{e,a}(O) - \underline{M}^{e,a}(O) \cdot \underline{u} \right)$$

→ il momento di reazione ha solo la parte statica

Per finire, $\underline{F}_O^{z}$ sono determinate dalla prima equazione cardinale della dinamica

<u>Applicazione</u>: sistemi ad un grado di libertà con forze dipendenti solo dalle configurazioni

1 grado di libertà → 1 coordinata libera

Per ipotesi l'unica forza generalizzata dipende da q $\Rightarrow$ possiamo definire una energia potenziale

$$V = -\int Q(q)\,dq \quad \Rightarrow \quad Q = -\frac{dV}{dq}$$

Eq. di Lagrange conservative:

$$\mathcal{L} = K - V$$

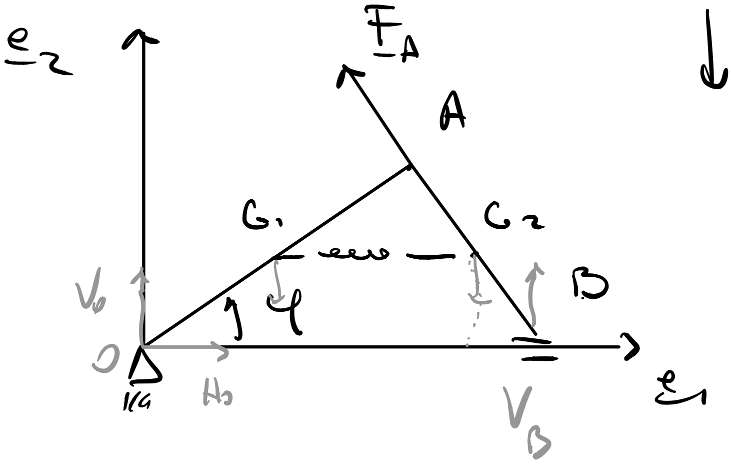
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0$$

Energia: $E = K + V$

Sistemi conservativi:

$$E = K + V = (K + V)\big|_{t=0}$$

Esempio



$\downarrow \underline{g}$

aste omogenee di lunghezza l massa m piano verticale

$\underline{F}_A$

Equazioni del moto $\quad K = K_{OA} + K_{AB}$

- K_{OA} moto intorno ad O

$$K_{OA} = \frac{1}{2} I_{3,O} \dot{\varphi}^2 = \frac{1}{2} \frac{m l^2}{3} \dot{\varphi}^2$$

- K_{AB} moto generico

$$K_{AB} = \frac{1}{2} m v_{G_2}^2 + \frac{1}{2} \frac{m l^2}{12} \dot{\varphi}^2$$

calcolare $\underline{v}_{G_2}$

$$\underline{x}_{G_2}(t) = \frac{3}{2} l \cos\varphi(t) \underline{e}_1 + \frac{l}{2} \sin\varphi(t) \underline{e}_2$$

$$\underline{v}_G = \frac{d}{dt} \underline{x}(t) \longrightarrow \underline{v}_G^2 = \underline{v}_G \cdot \underline{v}_G$$

Risultato

$$K = \frac{1}{2} m l^2 \dot{\varphi}^2 \left(\frac{2}{3} + 2 \sin^2\varphi \right)$$

Per $\mathcal{L} = K - V$, dobbiamo trovare V

$$V_{peso} = 2 m g \frac{l}{2} \sin\varphi = m g l \sin\varphi$$

$$V_{molle} = \frac{c}{2} l^2 \cos^2\varphi$$

$$V_{follower} = - \int F_A \underbrace{l \, d\varphi}_{\text{spostamento di } A} = - F_A l \varphi$$

Mettiamo tutto insieme

$$\mathcal{L} = K - V = \frac{1}{2} m \ell^2 \dot{\varphi}^2 \left(\frac{2}{3} + 2 \sin^2 \varphi \right)$$

$$- \left(m g \ell \sin\varphi + \frac{c}{2} \ell^2 \cos^2\varphi - F_A \ell \varphi \right)$$

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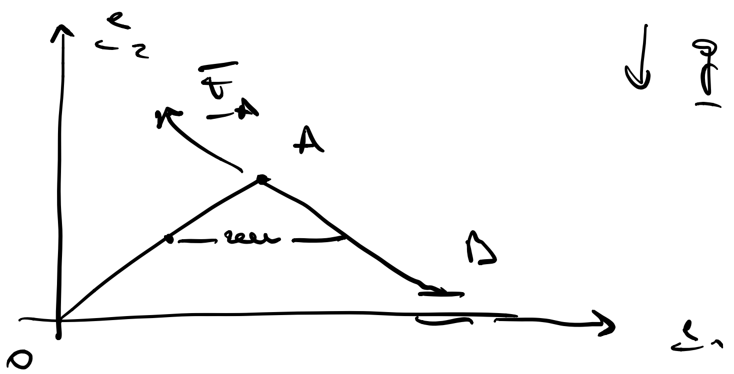
- TERZA PARTE



$\downarrow \underline{g}$ OA , AB

Equazioni del moto

- determinare $\mathcal{L} = K - V$
- $$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{L}}{\partial q} = 0$$

$$\mathcal{L} = \frac{1}{2} m \ell^2 \dot{\varphi}^2 \left(\frac{2}{3} + 2 \sin^2 \varphi \right) +$$

$$-\left(mg\ell \sin\varphi + \frac{c}{2}\ell^2\cos^2\varphi - F_A \ell\varphi \right)$$

- $$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial\dot\varphi}\right) = \frac{d}{d\tau}\left[m\ell^2\dot\varphi\left(\frac{2}{3} + 2\sin^2\varphi\right)\right]$$

$$= m\ell^2\ddot\varphi\left(\frac{2}{3} + 2\sin^2\varphi\right) + m\ell^2\dot\varphi\, 4\sin\varphi\cos\varphi\,\dot\varphi$$

- $$\frac{\partial\mathcal{L}}{\partial\varphi} = \frac{1}{2}m\ell^2\dot\varphi^2\, 2\cdot 2\sin\varphi\cos\varphi +$$

$$- mg\ell\cos\varphi + \frac{c}{2}\ell^2\, 2\cos\varphi\sin\varphi + F_A\ell$$

$$\frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot\varphi}\right) - \frac{\partial\mathcal{L}}{\partial\varphi} = m\ell^2\ddot\varphi\left(\frac{2}{3} + 2\sin^2\varphi\right)$$

$$+ m\ell^2\dot\varphi\, 4\sin\varphi\cos\varphi\,\dot\varphi - 2m\ell^2\dot\varphi^2\sin\varphi\cos\varphi$$

$$+ mg\ell\cos\varphi - c\ell^2\cos\varphi\sin\varphi - F_A\ell$$

$$= \boxed{m\ell^2\ddot\varphi\left(\frac{2}{3} + 2\sin^2\varphi\right) + 2m\ell^2\dot\varphi^2\sin\varphi\cos\varphi + mg\ell\cos\varphi - c\ell^2\cos\varphi\sin\varphi - F_A\ell = 0}$$

è l'equazione del moto

$$\boxed{\frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot q} - \frac{\partial\mathcal{L}}{\partial q} = 0} \longleftarrow \mathcal{L} = K - V$$

Conservazione dell' energia $E = K + V$

$(\mathcal{L} = K - V)$

$$K + V = \frac{1}{2} m \ell^2 \dot{\varphi}^2 \left(\frac{2}{3} + 2 \sin^2 \varphi \right) +$$

$$+ \left(m g \ell \sin\varphi + \frac{c}{2} \ell^2 \cos^2\varphi - F_A \, l \, \varphi \right)$$

$$= E = (K + V)_{t=0}$$

Se a $T=0$, sappiamo $\varphi(0) = \varphi_0$, $\dot{\varphi}(0) = \omega_0$

$\rightarrow$ allora conosciamo il valore di E

$$E = \frac{1}{2} m \ell^2 \omega_0^2 \left(\frac{2}{3} + 2 \sin^2\varphi_0 \right) + m g \ell \sin\varphi_0$$

$$+ \frac{c}{2} \ell^2 \cos^2\varphi_0 - F_A \, \ell \, \varphi_0$$

$$V(\varphi) = m g \ell \sin\varphi + \frac{c}{2} \ell^2 \cos^2\varphi - F_A \, \ell$$

$$E = \frac{1}{2} m \ell^2 \omega_0^2 \left(\frac{2}{3} + 2 \sin^2\varphi_0 \right) + V(\varphi_0) \qquad T=0$$

$$= \frac{1}{2} m \ell^2 \dot{\varphi}^2 \left(\frac{2}{3} + 2 \sin^2\varphi \right) + V(\varphi) \qquad T$$

ci permette di ricavare $\dot{\varphi}^2$ in funzione di φ

$$\dot{\varphi}^2 = \frac{2}{m\ell^2\left(\frac{2}{3}+2\sin^2\varphi\right)}\left(E - V(\varphi)\right) =: f(\varphi)$$

$$\downarrow$$

$$\ddot{\varphi} = \frac{1}{2m\ell^2\left(\frac{2}{3}+2\sin^2\varphi\right)}\Big[-2m\ell^2 f(\varphi)\sin\varphi\cos\varphi$$

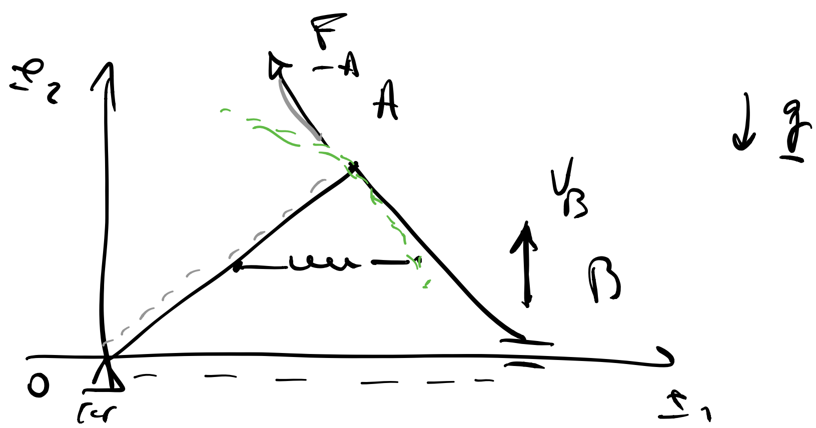
$$- mg\ell\cos\varphi + c\ell^2\sin\varphi\cos\varphi + F_A\,\ell\Big]$$

$$=: g(\varphi)$$

- Calcolare la reazione vincolare in B

$$\underline{M}^e(O) = \frac{d}{dt}\underline{L}(O)$$

Equazione cardinale della dinamica



$$\underline{L}(O) = \underline{L}_{OA}(O) + \underline{L}_{AB}(O)$$

- $O \in OA$ $\quad \underline{L}^{OA}(O) = I_{3,O}\,\underline{\omega} = \frac{m\ell^2}{3}\dot{\varphi}\,\underline{e}_3$

- $O \notin AB$ $\quad \underline{L}^{AB}(O) = \underline{L}^{AB}(G_2)$

$$+ (\underline{x}_{G_2} - \underline{x}_O)\wedge m\,\underline{v}_{G_2}$$

dove $\underline{L}^{AB}(G_2) = I_{3,0}(-\underline{\omega}) = -\frac{ml^2}{12}\dot{\varphi}\,\underline{e}_3$

$$\underline{L}(O) = ml^2\dot{\varphi}\,\underline{e}_3$$

$$\underline{M}(O) = V_B\, 2l\cos\varphi + F_A\, l +$$
$$- mg\frac{l}{2}\cos\varphi - mg\frac{3}{2}l\cos\varphi$$

$$\underline{M}(O) = \frac{d}{dt}\underline{L}(O) = ml^2\ddot{\varphi}\,\underline{e}_3$$

$$\Rightarrow \quad V_B\, 2l\cos\varphi + F_A\, l - mg\frac{l}{2}\cos\varphi - mg\frac{3}{2}l\cos\varphi$$
$$= ml^2\ddot{\varphi}$$

$$V_B = \underbrace{\frac{ml^2\ddot{\varphi}}{2l\cos\varphi}}_{\text{parte dinamica}} + \underbrace{\frac{1}{2l\cos\varphi}\left(-F_A l + 2mg\,l\cos\varphi\right)}_{\text{statica}}$$

$$= \frac{ml}{2\cos\varphi}\,g(\varphi) + \frac{1}{2l\cos\varphi}\left(-F_A l + 2mg\,l\cos\varphi\right)$$

V_B esplicitamente per qualsiasi φ, senza aver integrato le equazioni del moto