MECCANICA RAZIONALE

– LEZIONE DEL 15 APRILE 2020

– PRIMA PARTE

Problema della dinamica

- Equazioni di Lagrange

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \qquad \forall\, i = 1, \dots, \ell$$

$$\mathcal{L} = K - V$$

$V \longrightarrow$ statica

K : (corpo rigido piano)

$\rightarrow$ generico $\qquad K = \frac{1}{2} M \underline{v_G^2} + \frac{1}{2} I_{3,G}\, \omega^2$

$\rightarrow$ pto O fisso $\qquad K = \frac{1}{2} I_{3,O}\, \omega^2$

$\rightarrow$ traslazione $\qquad K = \frac{1}{2} M \underline{v_G^2}$

$$\underline{x}_G \longrightarrow \frac{d}{dt}\underline{x}_G = \underline{v}_G \longrightarrow v_G^2$$

- Equazioni cardinali della dinamica

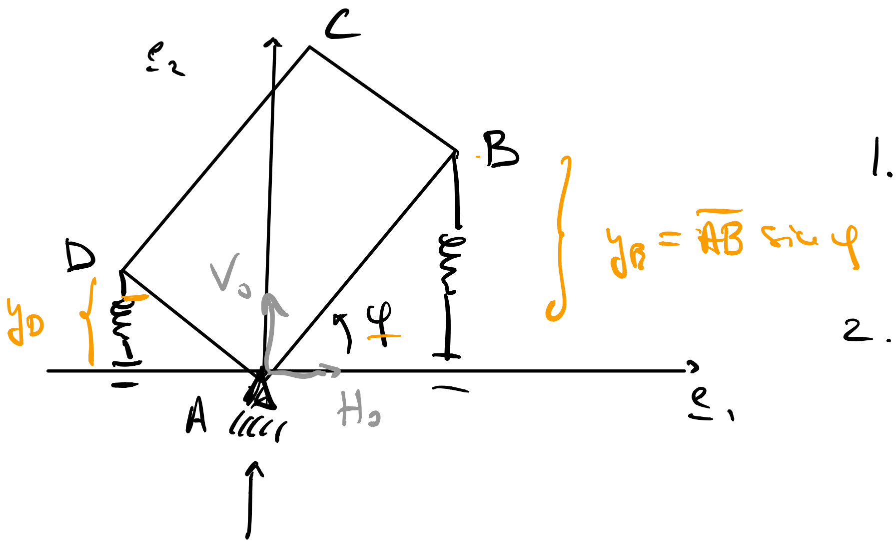
$$\underline{R}^e = \frac{d}{dt}\underline{P} \qquad \underline{P} = M\,\underline{v}_G$$

$$\underline{M}^e(O) = \frac{d}{dt}\underline{L}(O) + \underline{v}_O \wedge \underline{P}$$

$(= 0 \text{ se } O = G)$

$$\underbrace{\underline{L}(O) = I_O(\underline{\omega})}_{\text{se } O \in R} + M(\underline{x}_G - \underline{x}_O) \wedge \underline{v}_O$$

<u>Esercizio</u>



Piano orizzontale
rettangolo omogeneo
$\overline{AD} = 2\ell \qquad \overline{BC} = \ell$

1. Scrivere le eq. di moto

2. Condizioni iniziali $\varphi(0) = \frac{\pi}{4}$, $\dot{\varphi}(0) = \omega_0$
Scrivere la conservazione dell'energia

3. Calcolare le reazioni vincolari in A in funzione di φ, con le condizioni iniziali assegnate.

1) Eq. del moto : $\mathcal{L} = K - V$

$$K = \frac{1}{2} I_{A,3} \dot{\varphi}^2 = \frac{1}{2} \frac{m}{3} (\ell^2 + 4\ell^2) \dot{\varphi}^2$$

$$I_{33} = I_{11} + I_{22}$$

$$\frac{m}{3} a^2 , \frac{m}{3} b^2$$

$$I = \begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{12} & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix}$$

$$= \frac{1}{2} \frac{5}{3} m \ell^2 \dot{\varphi}^2$$

$$V = \frac{c}{2} y_B^2 + \frac{c}{2} y_D^2$$

$$= \frac{c}{2} (2\ell)^2 \sin^2\varphi + \frac{c}{2} \ell^2 \cos^2\varphi$$

$\sin^2 + \cos^2 = 1$

$$= \frac{c}{2} \ell^2 (3 \sin^2\varphi + 1)$$

Allora $\Rightarrow$ $\mathcal{L} = K - V = \frac{1}{2} \left(\frac{5}{3} m \ell^2 \dot{\varphi}^2 \right) +$

$$- \frac{c}{2} \ell^2 \left(3 \sin^2\varphi + 1 \right)$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

$$\rightarrow \quad \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \frac{1}{2} \frac{5}{3} m \ell^2 \, 2 \dot{\varphi}$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \frac{5}{3} m \ell^2 \ddot{\varphi}$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{\partial}{\partial \varphi}(K - V) = - \frac{\partial V}{\partial \varphi}$$

equilibrio? stabilità?

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\varphi} - \frac{\partial \mathcal{L}}{\partial \varphi} = \frac{5}{3} m \ell^2 \ddot\varphi - \frac{\partial}{\partial \varphi}(-V)$$

$$= \boxed{\frac{5}{3} m \ell^2 \ddot\varphi + 3\frac{c}{2}\ell^2\, 2 \sin\varphi \cos\varphi = 0}$$

2) Energia : $E = K + V$

$$= \frac{1}{2}\left(\frac{5}{3} m \ell^2 \dot\varphi^2\right) + \frac{c}{2}\ell^2\left(3\sin^2\varphi + 1\right)$$

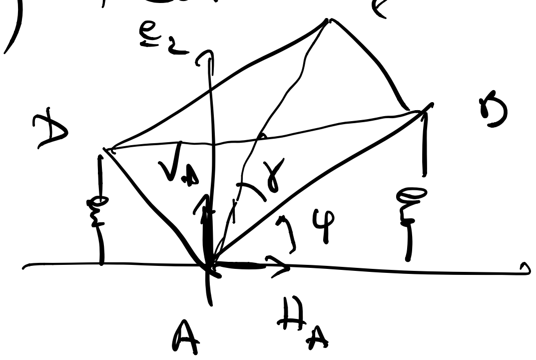
funzione del tempo

$$= \frac{1}{2}\left(\frac{5}{3} m \ell^2 \omega_0^2\right) + \frac{c\ell^2}{2}\left(\frac{3}{2} + 1\right) = E_{t=0}$$

$\dot\varphi = \omega_0$ $\qquad \varphi = \frac{\pi}{4}$

calcolata a $t=0$

3) Reazioni vincolari H_A, V_A

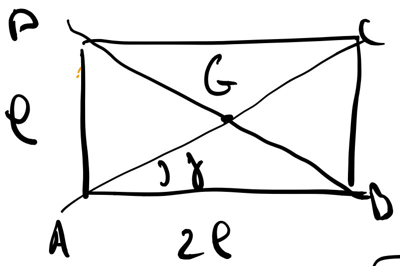


Equazioni Cardinali della Dinamica

$$\underline{R}^e = \dot{\underline{P}}$$

$$\underline{R}^e = H_A \underline{e}_1 + V_A \underline{e}_2 - c y_B \underline{e}_2 - c y_D \underline{e}_2$$

$$\Rightarrow \begin{cases} H_A = m \ddot x_G \\ V_A = c\, 2\ell \sin\varphi + c\,\ell \cos\varphi + m \ddot y_G \end{cases}$$



$$x_G = \overline{AG}\cos(\gamma+\varphi)$$
$$y_G = \overline{AG}\sin(\gamma+\varphi)$$

$$AC = \sqrt{\ell^2 + 4\ell^2} = \ell\sqrt{5}$$
$$AG = \frac{1}{2}\sqrt{\ell^2+4\ell^2} = \frac{\ell}{2}\sqrt{5}$$

$$\sin\gamma = \frac{BC}{AC} = \frac{1}{\sqrt{5}}\ , \qquad \cos\gamma = \frac{AB}{AC} = \frac{2}{\sqrt{5}}$$

Troviamo $\ddot{x}_G$, $\ddot{y}_G$

$$\dot{x}_G = -\overline{AG}\sin(\gamma+\varphi(t))\,\dot{\varphi}(t)$$
$$\dot{y}_G = \overline{AG}\cos(\gamma+\varphi(t))\,\dot{\varphi}(t)$$

$$\ddot{x}_G = -\overline{AG}\sin(\gamma+\varphi(t))\,\ddot{\varphi}(t) + \\ - \overline{AG}\cos(\gamma+\varphi(t))\,\dot{\varphi}^2(t)$$

$$\ddot{y}_G = \overline{AG}\cos(\gamma+\varphi(t))\,\ddot{\varphi}(t) + \\ - \overline{AG}\sin(\gamma+\varphi(t))\,\dot{\varphi}^2(t)$$

Mettiamo tutto insieme:

eq. del moto: $\ddot{\varphi} = -\frac{9}{5}\frac{c}{m}\sin\varphi\cos\varphi =: g(\varphi)$

cons. energia: $\dot{\varphi}^2 = \frac{6}{5}\frac{1}{m\ell^2}\left[E - \frac{c}{2}\ell^2(3\sin^2\varphi+1)\right] =: f(\varphi)$

$$H_A = m \ddot{x}_G = -\overline{AG} \sin(\gamma + \varphi(t))\, g(\varphi) +$$

$$- \overline{AG}\, m \cos(\gamma + \varphi(t))\, f(\varphi)$$

$V_A = \text{-----}$

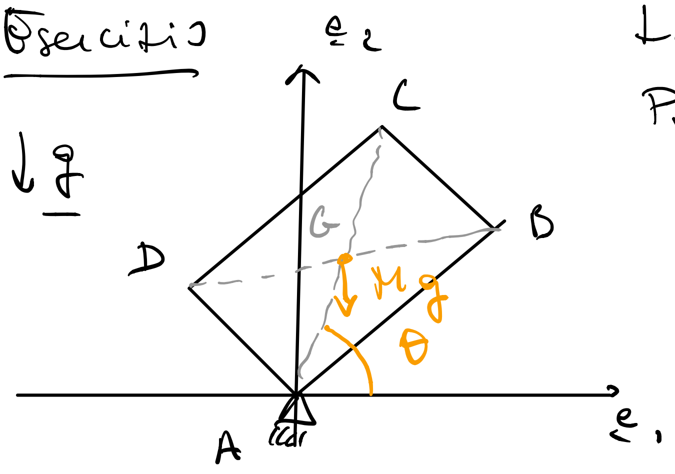
→ Abbiamo trovato le reazioni vincolari in funzione di φ

MECCANICA RAZIONALE

– LEZIONE DEL 15 APRILE 2020

– SECONDA PARTE

Esercizio



Lamina rettangolare omogenea. Piano verticale. Calcoliamo

- equazioni del moto
- reazioni in A per $\theta = -\frac{\pi}{2}$, con condizioni iniziali $\theta(0) = 0$, $\dot{\theta}(0) = 0$

Punto # 1 → calcoliamo $\mathcal{L}$.

$$\mathcal{L} = K - V$$

$$K = \frac{1}{2} I_{3,A} \dot{\theta}^2 = \frac{1}{2}\left(\frac{5}{3} m l^2\right)\dot{\theta}^2$$

$$V = mg \overline{AG} \sin\theta = mg \frac{\sqrt{5}}{2} l \sin\theta$$

$$\rightarrow \mathcal{L} = K - V =$$

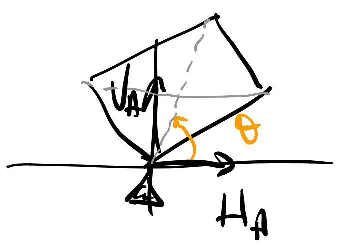
$$\overset{!}{=} \frac{1}{2}\left(\frac{5}{3} m l^2\right)\dot{\theta}^2 - mg\frac{\sqrt{5}}{2} l \sin\theta$$

Eq. di moto

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left(\frac{1}{2}\frac{5}{3} m l^2 \, 2\dot{\theta}\right) + mg\frac{\sqrt{5}}{2} l \cos\theta$$

$$\boxed{= \frac{5}{3} m l^2 \ddot{\theta} + mg\frac{\sqrt{5}}{2} l \cos\theta = 0}$$

<u>Punto #2</u>



$$\underline{R}^e = H_A \underline{e}_1 + V_A \underline{e}_2 - mg\,\underline{e}_2 = \underline{\dot{P}}$$

$$\hookrightarrow \boxed{\begin{cases} H_A = m\ddot{x}_G \\ V_A - mg = m\ddot{y}_G \end{cases}}$$

$$\begin{cases} x_G = \overline{AG}\cos\theta(t) \\ y_G = \overline{AG}\sin\theta(t) \end{cases} \Rightarrow \begin{cases} \dot{x}_G = -\overline{AG}\sin\theta(t)\,\dot{\theta}(t) \\ \dot{y}_G = \overline{AG}\cos\theta(t)\,\dot{\theta}(t) \end{cases}$$

$$\Rightarrow \begin{cases} \ddot{x}_G = -\overline{AG}\cos\theta(t)\,\dot{\theta}^2(t) - \overline{AG}\sin\theta(t)\,\ddot{\theta}(t) \\ \ddot{y}_G = -\overline{AG}\sin\theta(t)\,\dot{\theta}^2(t) + \overline{AG}\cos\theta(t)\,\ddot{\theta}(t) \end{cases}$$

Come primo, eq. del moto ∮ conservazione dell'energia

$$E = K + V = \frac{1}{2}\left(\frac{5}{3} m \ell^2 \dot{\theta}^2\right) + mg \frac{\ell\sqrt{5}}{2} \sin\theta$$

condizioni iniziali $\theta(0) = 0$

$\dot{\theta}(0) = 0$

$$= 0$$

$$\longrightarrow \quad \dot{\theta}^2 = - \frac{mg \frac{\ell\sqrt{5}}{2} \sin\theta}{\frac{1}{2}\frac{5}{3} m \ell^2} = - \frac{g}{\ell}\frac{3}{\sqrt{5}} \sin\theta$$

dall' eq. di moto

$$\ddot{\theta} = - \frac{3}{10}\sqrt{5}\,\frac{g}{\ell} \cos\theta$$

Andiamo a vedere $\theta = -\frac{\pi}{2}$

$$\dot{\theta}^2 = - \frac{g}{\ell}\frac{3}{\sqrt{5}} \sin\theta \Big|_{\theta = -\frac{\pi}{2}} = \underline{\underline{\frac{3}{\sqrt{5}}\frac{g}{\ell}}}$$

$$\ddot{\theta} = - \frac{3}{10}\sqrt{5}\,\frac{g}{\ell} \cos\theta \Big|_{\theta = -\frac{\pi}{2}} = 0$$

Andiamo a vedere $\ddot{x}_G$, $\ddot{y}_G$

$$\ddot{x}_G = -\overline{AG}\cos\theta(t)\,\dot{\theta}(t)^2 - \overline{AG}\sin\theta(t)\,\underbrace{\ddot{\theta}(t)}_{=0}$$

$$\xrightarrow{\theta=-\frac{\pi}{2}} 0$$

$$\ddot{y}_G = -\overline{AG}\sin\theta(t)\,\dot{\theta}(t)^2 + \overline{AG}\cos\theta(t)\,\underbrace{\ddot{\theta}(t)}_{=0}$$

$$\xrightarrow{\theta=-\frac{\pi}{2}} -\left(\frac{\ell}{2}\sqrt{3}\right)(-1)\left(\frac{3}{\sqrt{3}}\frac{g}{\ell}\right) = \frac{3}{2}g$$

Sostituiamo nell'espressione delle reazioni vincolari

$$\begin{cases} H_A = 0 \\ V_A = \underbrace{mg}_{\text{statico}} + \underbrace{m\frac{3}{2}g}_{\text{dinamico}} = \frac{5}{2}mg \end{cases}$$

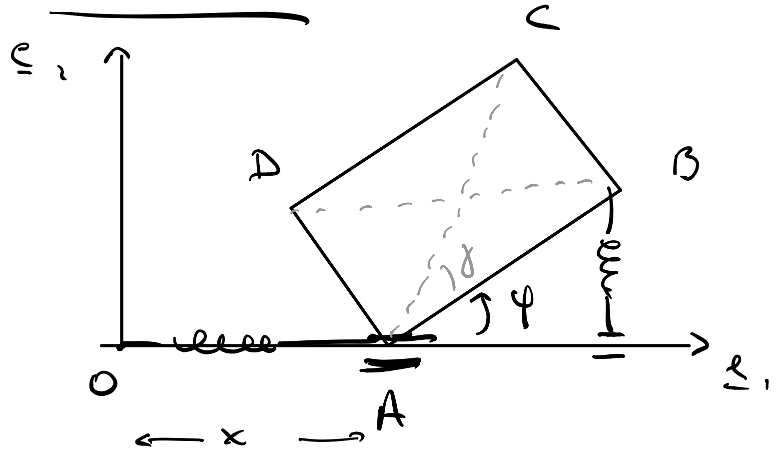
$$\left[\ \text{statico} \quad V = mg\,\overline{AG}\sin\theta \rightarrow \underbrace{V' = mg\,AC\cos\theta}_{=0} \quad \theta = \pm\frac{\pi}{2}\ \right]$$

MECCANICA RAZIONALE

- LEZIONE DEL 15 APRILE 2020

- TERZA PARTE

Esercizio



Piano orizzontale

Coordinate libere
(x, φ)

Molle : c

1) Eq. del moto

2) Reazioni in A lungo il moto

Punto 1)

Il moto è rototraslatorio

$$K = \frac{1}{2} m v_G^2 + \frac{1}{2} I_{3,G} \, \omega^2$$

$$\underline{\omega} = \dot{\varphi} \, \underline{e}_3$$

$$I_{G,3} = M \frac{\ell^2 + 4\ell^2}{12} = M \frac{5}{12} \ell^2$$

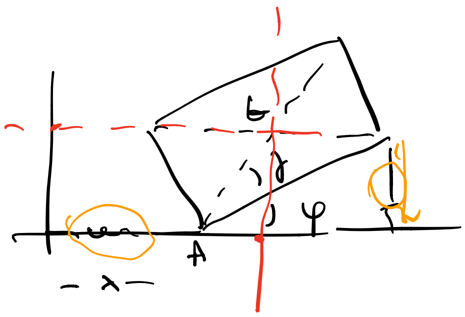
T H-S : $I_{z,A} = I_{z,G} + M \,(\text{distanza})^2$

$$\longrightarrow I_{z,G} = I_{z,A} - M \,(\text{distanza})^2$$

$$= M\ell^2 \frac{5}{3} - M\left(\ell^2 + \frac{\ell^2}{4}\right)$$

$$\underline{v}_G = \frac{d}{dt} x_G$$

$$\underline{x}_G = \left[x(t) + \overline{AG}\cos(\gamma + \varphi(t)) \right] \underline{e}_1 + \left[\overline{AG}\sin(\gamma + \varphi(t)) \right] \underline{e}_2$$



$$\overline{AG} = \frac{\ell}{2}\sqrt{5}$$

$$\sin\gamma = \frac{1}{\sqrt{5}}, \quad \cos\gamma = \frac{2}{\sqrt{5}}$$

$$\frac{d}{dt}\underline{x}_G = \left[\dot{x}(t) - \overline{AG}\sin(\gamma+\varphi)\dot{\varphi} \right] \underline{e}_1 + AG\cos(\gamma+\varphi)\dot{\varphi}\,\underline{e}_2$$

Calcolo di $K \longrightarrow \|\underline{v}_G\|^2 = \underline{v}_G \cdot \underline{v}_G \qquad \underline{v}_G = \frac{d}{dt}\underline{x}_G$

Siccome $\underline{e}_1 \cdot \underline{e}_2 = 0$

$$\|\underline{v}_G\|^2 = \left(\dot{x}(t) - \overline{AG}\sin(\gamma+\varphi)\dot{\varphi}\right)^2 + \left(\overline{AG}\cos(\gamma+\varphi)\dot{\varphi}\right)^2$$

$$= \dot{x}(t)^2 - 2\dot{x}\dot{\varphi}\,\overline{AG}\sin(\gamma+\varphi) + \overline{AG}^2\dot{\varphi}^2$$

Quindi

$$K = \frac{1}{2} m \|\underline{v}_G\|^2 + \frac{1}{2} I_{G,3}\,\omega^2 =$$

$$= \frac{1}{2} m \left(\dot{x}^2 + \dot{\varphi}^2\left(\frac{5}{4}\ell^2 + \frac{5}{12}\ell^2\right) - \ell\sqrt{5}\,\dot{x}\dot{\varphi}\sin(\gamma+\varphi) \right)$$

Potenziale

$$V = \frac{c}{2}x^2 + \frac{c}{2}(2\ell\sin\varphi)^2 = \frac{c}{2}x^2 + 2c\ell^2\sin^2\varphi$$

$$\mathcal{L} = \frac{1}{2} m \left(\dot{x}^2 + \dot{\varphi}^2 \frac{5}{3} \ell^2 - \ell \sqrt{5}\, \dot{x} \dot{\varphi} \sin(\gamma + \varphi) \right)$$
$$- \frac{c}{2} x^2 + 2c\ell^2 \sin^2\varphi$$

Equazioni del moto $\rightarrow$ per x e per φ

$x \rfloor$
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\left(\frac{1}{2} m \dot{x} - \frac{1}{2} m \ell \sqrt{5}\, \dot{\varphi} \sin(\gamma+\varphi) \right)$$
$$+ cx$$
$$\boxed{= \frac{1}{2} m \ddot{x} - \frac{1}{2} m \ell \sqrt{5}\, \ddot{\varphi} \sin(\gamma+\varphi) + - \frac{1}{2} m \ell \sqrt{5}\, \dot{\varphi}^2 \cos(\gamma+\varphi) + cx = 0}$$

$\varphi \rfloor$
$$\mathcal{L} = \frac{1}{2} m \left(\dot{x}^2 + \dot{\varphi}^2 \frac{5}{3} \ell^2 - \ell \sqrt{5}\, \dot{x} \dot{\varphi} \sin(\gamma + \varphi) \right)$$
$$- \frac{c}{2} x^2 + 2c\ell^2 \sin^2\varphi$$
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) - \frac{\partial \mathcal{L}}{\partial \varphi} = \frac{d}{dt}\left(\dot{\varphi} \frac{5}{3} m \ell^2 - \frac{1}{2} m \ell \sqrt{5}\, \dot{x} \sin(\gamma+\varphi) \right)$$
$$- \left(- \frac{1}{2} m \ell \sqrt{5}\, \dot{x} \dot{\varphi} \cos(\gamma+\varphi) - 4 c \ell^2 \cos\varphi \sin\varphi \right)$$
$$= \frac{5}{3} m \ell^2 \ddot{\varphi} - \frac{1}{2} m \ell \sqrt{5}\, \ddot{x} \sin(\gamma+\varphi) +$$

$$-\frac{1}{2} m l \sqrt{5}\, \dot{x}\dot{\varphi} \cos(\gamma+\varphi)$$

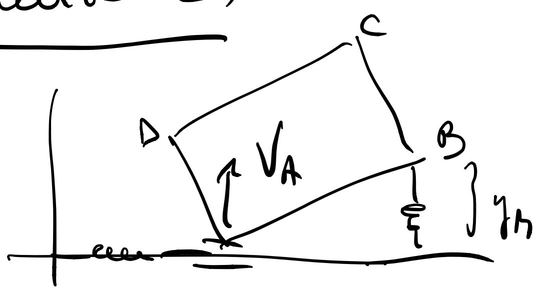
$$+\frac{1}{2} m l \sqrt{5}\, \dot{x}\dot{\varphi} \cos(\gamma+\varphi) + 4cl^2\cos\varphi\sin\varphi$$

$$= \boxed{\frac{5}{3} m l^2 \ddot{\varphi} - \frac{1}{2} m l \sqrt{5}\, \ddot{x} \sin(\gamma+\varphi) + 4cl^2\cos\varphi\sin\varphi = 0}$$

Note: eq. di conservazione dell'energia $E = K + V$ vale ancora, però è una sola, per cui non possiamo ricavare ad esempio $\dot{\varphi}^2$ in funzione di x e φ: dipenderà anche da $\dot{x}$

Quindi per sistemi a più gradi di libertà, per ottenere $\ddot{x}$ e $\ddot{\varphi}$ in funzione di x e φ, dobbiamo risolvere le equazioni del moto

Punto 2)



Equazione cardinale della dinamica lungo e_2

$$V_A - c\, y_B = V_A - 2\ell c \sin\varphi = m\, \ddot{y}_G$$

da cui

$$V_A = c\, 2\ell \sin\varphi + m \Big(\overline{AG} \cos(\gamma+\varphi)\, \ddot{\varphi} + - \overline{AG} \sin(\gamma+\varphi)\, \dot{\varphi}^2 \Big)$$

$\hookrightarrow AG = \frac{\ell}{2}\sqrt{5}$

Possiamo calcolare V_A in funzione di φ, risolvendo per $\varphi = \varphi(T)$