Cosmology 1

2019/2020
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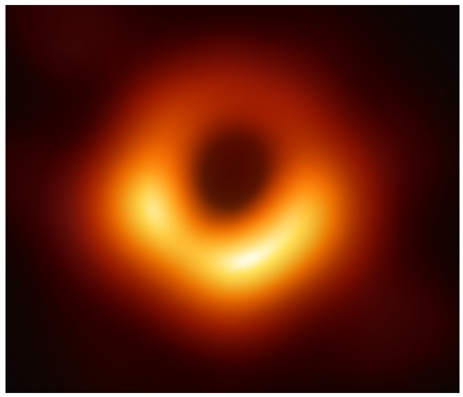
First intermediate test
Topic: general relativity.
Deadline: April 23, 13:00.



One year ago the Event Horizon Telescope (EHT) collaboration published the first image of the event horizon of a black hole (BH), M87* at the center of the galaxy M87, located at $D = 16.8$ Mpc (1 pc $= 3.09 \times 10^{16}$ m, 1 Mpc is one million pc), that is known from previous measurements to be as massive as $M = 6.5 \times 10^9$ solar masses $M_\odot$ (1 $M_\odot = 1.99 \times 10^{30}$ kg). The angular resolution of the image is 25 μas (micro-arcseconds). We can think of this image as to a disc of gas that is rotating around the BH almost face-on.

(1) Assuming Euclidean geometry, what is the angular extent of the gravitational radius $R_G = GM/c^2$ as seen from the Earth? would it be possible to image it with the nominal EHT angular resolution? To have an idea of how big it is, compare R_G with the size of the Solar System, then compute the radius of a sphere on the surface of the moon that subtends the same angle as seen from the Earth. These two facts give a feeling of how big this BH is and how far this "nearby" galaxy is.

The angular diameter of the imaged ring is indeed 42 μas. To understand how curvature distorts photon paths we can compute the so-called photon capture radius of a BH, following these steps.

(2) Let's first write down an effective potential for a photon in the Newtonian case. Assuming that b is the photon impact parameter for a point mass M at the center of a cartesian coordinate system in which the photon trajectory lies at $z = 0$ and is aligned with the x-axis, call $x = ct$ its x-coordinate, r its distance from the mass M, and demonstrate that its equation of motion can be written as:

$$\left(\frac{dr}{dt}\right)^2 = c^2 - V_{\rm eff}(r) \quad (1)$$

Find the expression for the effective potential $V_{\rm eff}$ and interpret it.

(3) Now (using again $c = 1$) write the conserved quantities as $p_0 = -\tilde{E}$ and $p_\varphi = \tilde{L}$, and considering the photon trajectory at very large distances identify $\tilde{E} = h\nu_\infty$ as the photon energy at infinity and $\tilde{L} = b\tilde{E}$ as the photon angular momentum, b being its impact parameter. (*Hint:* let the photon travel at negative y values, so that its motion is counter-clockwise in the $x - y$ plane. Write the cartesian and spherical components of the photon momentum for a flat Minkowski spacetime and for a trajectory at $z = 0$ or $\vartheta = \pi/2$).

(4) Assuming a Schwartzschild metric, and calling λ an affine parameter for the null geodesic of a photon, compute $d^2r/d\lambda^2$ for the photon using the geodesic equation and the condition $d\vec{x}/d\lambda \cdot d\vec{x}/d\lambda = 0$. Find the condition for which the radial acceleration of the photon is positive or negative. How does this result compare with that obtained in class using the effective potential?

(5) From $\vec{p} \cdot \vec{p} = 0$ work out the equation of motion for the photon around the black hole. The photon can escape the black hole if the orbit's pericenter is such to avoid regions with $d^2r/d\lambda^2 < 0$. The pericenter can be computed as the point in which $dr/d\lambda$ changes sign, thus passing through 0. Using the espression for $(dr/d\lambda)^2$ obtain a relation between the orbit pericenter $r_{\rm min}$ and the impact parameter b. Now impose the escape condition worked out in point (4) to obtain a critical impact parameter, called **photon capture radius**. Argue that geodesics can be traveled in both spatial directions in this static metric (outside the event horizon), so that the photon capture radius defines the effective apparent size of the BH event horizon, the BH "shadow".

(6) How does its angular extension for M87* compares with the observed ring?

This is only an order-of-magnitude estimate based on the assumption of a spherically-symmetric, i.e. non-rotating, BH; full ray-tracing of radiation emitted by gas orbiting around a rotating black hole is needed to get a proper prediction.