LESSON 12.

1. The language of categories.

Category theory was introduced by Samuel Eilenberg and Saunders Mac Lane in 1942-45 in their study of algebraic topology. They introduced the concepts of categories, functors, and natural transformations, with the goal of understanding the processes that preserve mathematical structure. In Algebraic Geometry it was much developed by Alexander Grothendieck, in his language of schemes.

Category theory has proven to be a powerful language for expressing some general facts and constructions that are encountered mainly in branches of algebra and geometry. Here we give an elementary introduction limiting ourselves to the simplest definitions and examples.

Definition 1.1. A category $\mathcal{C}$ consists of the following data:

(1) A class $ob(\mathcal{C})$ whose elements are called objects of the category;

(2) For each pair $A, B \in ob(\mathcal{C})$ of objects, a set indicated with $Hom_{\mathcal{C}}(A, B)$, or $\mathcal{C}(a, B)$, called set of morphisms or arrows from A to B. Instead of writing $f \in Hom_{\mathcal{C}}(A, B)$ it is common to use $f : A \to B$.

(3) For each triple of objects A, B, C a map of sets called composition:

$$Hom_{\mathcal{C}}(A, B) \times Hom_{\mathcal{C}}(B, C) \to Hom_{\mathcal{C}}(A, C),$$

such that

$$(f, g) \to g \circ f.$$

(4) For each object A a special element $1_A \in Hom_{\mathcal{C}}(A, A)$ called identity of A.

It is also assumed that the following axioms hold:

a) Composition is associative;

b) Identity acts as a neutral element for the composition (when it is defined).

The categories that are best known (but we will also meet others) are those in which we can interpret morphisms as particular functions between sets, their composition is the usual composition of functions, and the identity is the usual identity.

In particular we have:

(1) The category of sets, indicated with the symbol Set, in which $Hom(A, B) = Set(A, B)$ is the set of arbitrary maps from A to B.

(2) The category *Grp* of groups and homomorphisms between groups, *Ab* of abelian groups and group homomorphisms, *Rng* of rings and homomorphisms of rings, or Mod_R of modules on a ring R with homomorphisms of R-modules, etc.

(4) *Top* with objects the topological spaces and morphisms the continuous functions.

(5) The coverings of a given topological space and the covering maps.

(6) The notion of subcategory is rather natural: $\mathcal{C}'$ is a subcategory of $\mathcal{C}$ if the class $ob(\mathcal{C}')$ is contained in $ob(\mathcal{C})$ and, for any pair of objects A, B in $\mathcal{C}'$, $Hom_{\mathcal{C}'}(A, B) \subset Hom_{\mathcal{C}}(A, B)$. The subcategory is called full if equality holds: $Hom'_{\mathcal{C}}(A, B) = Hom_{\mathcal{C}}(A, B)$.

(7) A first example of a category where morphisms cannot be thought of as simple functions is that of a *poset.* i.e. a partially ordered set P. The objects are the elements of P and

$$Hom_P(a, b) = \begin{cases} \{*\} & \text{if } a \leq b; \\ \emptyset & \text{otherwise.} \end{cases}$$

Here $\{*\}$ denotes a set with only one element, the singleton. A particular case of a poset category is $Op(X)$, the category of the open subsets of a topological space X.

The second notion we are going to introduce formalizes the idea of transformation of categories.

Definition 1.2. A (covariant) functor $F : \mathcal{A} \to \mathcal{B}$ from the category $\mathcal{A}$ to the category $\mathcal{B}$ is a law that associates to every object X of $\mathcal{A}$ an object $F(X)$ of $\mathcal{B}$ and to every morphism $f : X \to Y$ in $\mathcal{A}$ a morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{B}$, in such a way that
a) $F(f \circ g) = F(f) \circ F(g)$ (when the composition is defined),
b) $F(1_X) = 1_{F(X)}$.

The composition of functors can be done as in the case of functions.

Contravariant functors are defined by imposing that to every morphism $f : X \to Y$ is associated a morphism $F(f) : F(Y) \to F(X)$ so that we have $F(f \circ g) = F(g) \circ F(f)$. In other words, contravariant functors invert the arrows.

Given a category $\mathcal{C}$, we can define the opposite category $\mathcal{C}^0$, or $\mathcal{C}^{op}$, whose objects are the same as those of $\mathcal{C}$ while $Hom_{\mathcal{C}^0}(A, B) = Hom_{\mathcal{C}}(B, A)$. It is easily seen that a contravariant functor from $\mathcal{A}$ to $\mathcal{B}$ is also a covariant functor from $\mathcal{A}$ to $\mathcal{B}^0$ (or from $\mathcal{A}^0$ to $\mathcal{B}$).

Example 1.3. *Examples of functors.*

1. *Forgetful functors.* The law $U : Grp \to Set$ which maps a group to its underlying set and a group homomorphism to its underlying function of sets is a functor. Functors like this, which "forget" some structure, are termed forgetful functors. Another example is the

functor $Rng \to Ab$ which maps a ring to its underlying additive abelian group. Morphisms in Rng (ring homomorphisms) become morphisms in Ab (abelian group homomorphisms).

2. *Free functors.* Going in the opposite direction of forgetful functors are free functors. The free functor $F : Set \to Ab$ sends every set X to the free abelian group generated by X. Functions are mapped to group homomorphisms between free abelian groups.

3. *Representable functors.* Let $\mathcal{C}$ be a category. Each object $A \in ob(\mathcal{C})$ allows to define the following functor $h^A : \mathcal{C} \to Set$. For each object $X \in ob(\mathcal{C})$, $h^A(X) := Hom_{\mathcal{C}}(A, X) \in ob(Set)$. For each morphism $f : X \to Y$ in $\mathcal{C}$, we define $h^A(f) : Hom_{\mathcal{C}}(A, X) \to Hom_{\mathcal{C}}(A, Y)$ through the composition: $h^A(g) := f \circ g$. The functor h^A is usually denoted by $h^A := Hom_{\mathcal{C}}(A, -)$ and is a covariant functor which is said to be represented by the object A of $\mathcal{C}$.

In a completely analogous way we can define the contravariant functor $h_A := Hom_{\mathcal{C}}(-, A)$.

Among the categorical ideas there is that of isomorphism, which generalizes that of bijection between sets, of isomorphism of groups, of homeomorphism between topological spaces etc.

An isomorphism f between two objects A, B of a category $\mathcal{C}$ is a morphism $f : A \to B$ such that there is another morphism $g : B \to A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$.

The following property follows easily from the axioms of category.

Proposition 1.4. *(1) If $f : A \to B$ is an isomorphism, the morphism $g : B \to A$ such that $g \circ f = 1_A$, $f \circ g = 1_B$ is unique (and denoted f^{-1}).*

(2) If $f : A \to B$ is an isomorphism in $\mathcal{C}$ and $F : \mathcal{C} \to \mathcal{D}$ is a functor, then also $F(f) : F(A) \to F(B)$ is an isomorphism (in $\mathcal{D}$).

To complete the categorical approach it is convenient to introduce the last formal definition, the one that allows to treat the functors between two given categories $A \to B$ like the objects of a new category. To do this, we must define the morphisms between two such functors, which we will call natural transformations. We give the definition for covariant functors, the contravariant case is similar.

Definition 1.5. Given two functors $F, G : \mathcal{A} \to \mathcal{B}$ between two categories, a natural transformation $\varphi : F \to G$ between the two functors consists in giving, for each object $A \in ob(\mathcal{A})$ a morphism $\varphi_A : F(A) \to G(A)$ (in $\mathcal{B}$) such that, for each pair of objects $A, B \in ob(\mathcal{C})$ and for each morphism $f : A \to B$ the following diagram is commutative:

$$\begin{array}{ccc} F(A) & \xrightarrow{\varphi_A} & G(A) \\ \text{F(f)} \downarrow & & \downarrow \text{G(f)} \\ F(B) & \xrightarrow{\varphi_B} & G(B) \end{array}$$

The class of natural transformations between two functors $F, G : \mathcal{A} \to \mathcal{B}$ is denoted by $Nat(F, G)$. Often it is a set, which can therefore be taken as the set of morphisms to define the category of functors from category $\mathcal{A}$ to category $\mathcal{B}$. We will indicate with $F(\mathcal{A}, \mathcal{B})$ this category of functors. The properties of identity and composition are easy to verify.

From the general ideas, it follows the definition of natural isomorphism between two functors: it is a natural transformation that admits an inverse, and also that of **equivalence of categories.** An equivalence between the categories $\mathcal{A}, \mathcal{B}$ is a functor $F : \mathcal{A} \to \mathcal{B}$ satisying the following two conditions:

1. for any $Y \in ob(\mathcal{B})$ there exists $X \in ob(\mathcal{A})$ such that $Y \simeq F(X)$;
2. for any pair of objects A, B in $\mathcal{A}$, F gives a bijection $Hom(A, B) \xrightarrow{F} Hom(F(A), F(B))$.

We introduce a category $\mathcal{C}$ whose objects are the affine algebraic sets over a fixed algebraically closed field K and the morphisms are the regular maps. We consider also a second category $\mathcal{C}'$ with objects the K-algebras and morphisms the K-homomorphisms. Then there is a contravariant functor that operates on the objects sending X to $\mathcal{O}(X) = K[X]$, and on the morphisms sending φ to the associated comorphisms φ^*. Note that this functor can be interpreted as the representable functor $h_{\mathbb{A}^1}$.

If we restrict the class of objects of $\mathcal{C}'$ taking only the finitely generated reduced K-algebras (a full subcategory of the previous one), then this functor becomes an equivalence of categories. Indeed the construction of the comorphism establishes a bijection between the Hom sets $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $\mathrm{Hom}_{\mathcal{C}'}(\mathcal{O}(Y), \mathcal{O}(X))$. Moreover, for any finitely generated K-algebra A, there exists an affine algebraic set X such that A is K-isomorphic to $\mathcal{O}(X)$. To see this, we choose a finite set of generators of A, such that $A = K[\xi_1, \dots, \xi_n]$. Then we can consider the surjective K-homomorphism Ψ from the polynomial ring $K[x_1, \dots, x_n]$ to A sending x_i to ξ_i for any i. In view of the fundamental theorem of homomorphism, it follows that $A \simeq K[x_1, \dots, x_n]/\ker \Psi$. The assumption that A is reduced then implies that $X := V(\ker \Psi) \subset \mathbb{A}^n$ is an affine algebraic set with $I(X) = \ker \Psi$ and $A \simeq \mathcal{O}(X)$.

We note that changing system of generators for A changes the homomorphism Ψ, and by consequence also the algebraic set X, up to isomorphism. For instance let A be a polynomial ring in one variable t: if we choose only t as system of generators, we get $X = \mathbb{A}^1$, but if we choose t, t^2, t^3 we get the affine skew cubic in $\mathbb{A}^3$.

As a consequence of the previous discussion we have the following:

Corollary 1.6. *Let X, Y be affine varieties. Then $X \simeq Y$ if and only if $\mathcal{O}(X) \simeq \mathcal{O}(Y)$.*

We conclude this Lesson defining an important functor. Let X be a quasi-projective algebraic variety over a field K. We consider the category $Op(X)$ of the open subsets of X, interpreted as topological space with the Zariski topology. The second category is $K - alg$, the category of K-algebras and K-homomorphisms. We define a contravariant functor $\mathcal{O}_X : Op(X) \to K - alg$ such that, for any open subset $U \subset X$, $\mathcal{O}_X(U) = \mathcal{O}(U)$, the ring of regular functions on U interpreted as quasi-projective variety. Given a morphism in $Op(X)$, this is an inclusion $U \hookrightarrow V$; this is sent by the functor $\mathcal{O}_X$ to the natural restriction map $\mathcal{O}(V) \to \mathcal{O}(U)$.

$\mathcal{O}_X$ is called the sheaf of regular functions on the variety X.