Lecture 1
Introduction to decision theory

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Preferences and Utility function

Example:

Consider the following items:

Apple, Milk, Chocolate, Ice cream

What do you prefer?

Apple	vs	Milk	
Apple		Chocolate	
Apple		Ice cream	
Milk		Chocolate	
Milk		Ice cream	
Chocolate		Ice cream	

	comparisons			Preferred item	Notation
1	Apple	vs	Milk	Milk	Apple $\prec$ Milk
2	Apple		Chocolate	Chocolate	Apple $\prec$ Chocolate
3	Apple		Ice cream	Apple	Apple $\succ$ Ice cream
4	Milk		Chocolate	Indifferent	Milk $\sim$ Chocolate
5	Milk		Ice cream	I do not know	?
6	Chocolate		Ice cream	Ice cream	Chocolate $\prec$ Ice cream

Problems:

1. Not complete: # 5
2. Inconsistent: Apple, Chocolate, Ice cream

Chocolate $\succ$ Apple $\succ$ Ice cream $\succ$ Chocolate

2 # 3 # 6

<u>No transitive preferences</u>

	comparisons			Preferred item	Notation
1	Apple	vs	Milk	Milk	Apple $\prec$ Milk
2	Apple		Chocolate	Chocolate	Apple $\prec$ Chocolate
3	Apple		Ice cream	Ice cream	Apple $\prec$ Ice cream
4	Milk		Chocolate	Indifferent	Milk $\sim$ Chocolate
5	Milk		Ice cream	Ice cream	Milk $\prec$ Ice cream
6	Chocolate		Ice cream	Ice cream	Chocolate $\prec$ Ice cream

Preferences are:

1. Complete
2. Consistent: Ice cream $\succ$ Chocolate $\sim$ Milk $\succ$ Apple

6 # 4 # 1

<u>Transitive preferences</u>

Axioms of Rational Choice

- Completeness
  - for any pair of situations, A and B, an individual can always specify exactly one of these possibilities:
    - A is preferred to B
    - B is preferred to A
    - A and B are equally attractive
- Transitivity
  - For any triplet situations, A, B and C:
    - if A is preferred to B, and B is preferred to C, then A is preferred to C
  - assumes that the individual’s choices are internally consistent
- Continuity
  - if A is preferred to B, then situations suitably “close to” A must also be preferred to B
  - used to analyze individuals’ responses to relatively small changes in income and prices

When preferences are complete and transitive we say that **preferences are rational**.

Utility

- Given these assumptions, it is represent preferences using a function $u(x)$ where x is an element of the set X (set of items)
- In the previous example $\mathrm{X} = \{Apple,\ Milk,\ Chocolate,\ Ice\ cream\}$
- Economists call this utility function
- if A is preferred to B, then the utility assigned to A exceeds the utility assigned to B

$$u(A)\ > u(B)$$

Definition:

A function $u: X \rightarrow R$ is an utility function representing preferences on X if for each pair of items $x, y \in X$:

If $x \succcurlyeq y$ if and only if $u(x) \geq u(y)$

In the previous example:

Ice cream $\succ$ Chocolate $\sim$ Milk $\succ$ Apple

Function $u(x)$ represents these preferences if and only if

$$u(\textit{Ice cream}) > u(Chocolate) = u(Milk) > u(Apple)$$

i.e. for example $u(\textit{Ice cream}) = 4,\ u(Chocolate) = u(Milk) = 3,\ \ u(Apple) = 2$

Preferences can be represented by an utility function only if they are rational (necessary condition)

The reverse is not necessarily true: for example lexicographic preferences are rational but cannot be represented by an utility function

We need more assumptions:

For example either

1) a finite set X or
2) continuous preferences

Then

1. if preferences are rational and X has a finite number of elements, then preferences can be represented by an utility function
2. if preferences are rational and continuous, then preferences can be represented by an utility function

Example:

Consider a situation with two goods X and Y and let x and y be the respective quantities.

A basket is represented by the quantities of the two goods, i.e. (x, y)

Suppose a subject ranking the baskets only considering the quantity of good X. Comparing two baskets he considers the quantity of good Y only if the two baskets have the same quantity of X. For example:

$$(10, 20) \succ (9, 100) \text{ or } (10, 21) \succ (10, 19)$$

You can prove that these preferences are rational but are not continuous

Indeed consider the two bundles $(10, 21) \succ (10, 19)$. If you reduce the quantity x of the first bundle by an arbitrarily smaller quantity ε the preference relation is reversed. i.e. $(10 - \varepsilon, 21) \prec (10, 19)$ for any $\varepsilon > 0$.

Then these preferences cannot be represented by an utility function

Questions set 1,
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1) Four items, A, B, C, D. Assume $D \succ C$, $C \succ$ B, $B \succ$ A, $C \succ$ B, $B \succ$ A

Are these preferences satisfying completeness?

2) Four items, A, B, C, D. Assume $D \succ$ C, $C \succ$ B, $B \succ$ A, $C \succ$ B, $C \succ$ A, $B \succ$ A

Are these preferences satisfying transitivity?

3) Consider a situation with two goods X and Y and let x and y be the respective quantities. A basket is represented by the quantities of the two goods, i.e. (x, y). A subject cares only to the smaller quantity between x and y.

Check if these preferences are rational and continuous and state if can be represented by an utility function

Choice under risk

Up until now, we have thought of the objects that our decision makers are choosing as being physical items

However, we can also think of cases where the outcomes of the choices we make are uncertain - we don’t know exactly what will happen when we do a particular choice. For example:

- You are deciding whether or not to invest in a business
- You are deciding whether or not to go skiing next month
- You are deciding whether or not to buy a house that straddles the San Andreas fault line

In each case the outcomes are uncertain.

Here we are going to think about how to model a decision maker who is making such choices.

Economists tend to differentiate between two different types of ways in which we may not know for certain what will happen in the future: risk and ambiguity.

Risk: the probabilities of different outcomes are known, Ambiguity: the probabilities of different outcomes are unknown

Now we consider models of choice under risk,

An example of choice under risk

For an amount of money ***£ x***, you can flip a coin. If you get heads, you get ***£10***. If you get tails, you get ***£0***.

Assume there is a 50% chance of heads and a 50% chance of tails.

For what price ***x*** would you choose to play the game?

i.e. you have a choice between the following two options.

- 1. Not play the game and get nothing
- 2. Play the game, and get $-x$ for sure, plus a 50% chance of getting \$10.

Figure out the expected value (or average pay-out) of playing the game, and see if it is bigger than 0.

If it is bigger than 0, then you play the game, otherwise you don't play.

With a 50% chance you will get £$10 - x$,

With a 50% chance you will get $-x$.

Thus, the average payoff is:

$$0.5(10 - x) + 0.5(-x) = 5 - x$$

Thus the value of the game is £ $5 - x$.

you should play the game if the cost of playing is less than £ 5.

Lotteries (or prospects)

Decision making under risk can be considered as a process of choosing between different lotteries.

A lottery (or prospect) consists of a number of possible outcomes with their associated probability

It can be described as:

$$\boldsymbol{q} = (x_1, p_1; x_2, p_2; \dots x_n, p_n)$$

where

x_i represents the *i^th^* outcome and

p_i is its associated probability, $p_i \in [0,1]\ \forall i$ and $\sum_i p_i = 1$.

In the example the choice is between:

$$\boldsymbol{r} = (10 - x, 0.5; -x, 0.5)$$
$$\boldsymbol{s} = (0, 1)$$

in this last case we omit probability and we can write $\boldsymbol{s} = (0)$.

When an outcomes is for sure (i.e. its probability is 1) we write only the outcome.

$\boldsymbol{s} = (x)$ means that the outcome x is for sure

Sometime we can omit the zero outcomes, so the lottery $\boldsymbol{r} = (10, 0.5; 5, 0.3; 0, 0.2)$ can be written as $\boldsymbol{r} = (10, 0.5; 5, 0.3)$

Compound lotteries

Lotteries can be combined

From the previous example:

suppose you have the following lottery of lotteries:

$$\boldsymbol{c} = \left(\boldsymbol{r}, \frac{1}{2}; \boldsymbol{s}, \frac{1}{2}\right)$$

where

$\boldsymbol{r} = (10 - x, 0.5; -x, 0.5)$ and

$\boldsymbol{s} = (0, 1)$.

Then, the resulting lottery is:

$$\boldsymbol{c} = \left(10 - x, \frac{1}{4};\ -x, \frac{1}{4};\ 0, \frac{1}{2}\right)$$

More in general

Consider the two following lotteries

$\boldsymbol{r} = (x_1, p_1;\ \dots x_n, p_n)$ and

$\boldsymbol{s} = (y_1, q_1;\ \dots y_n, q_n)$,

then

$$\boldsymbol{c} = (\boldsymbol{r}, a; \boldsymbol{s}, 1 - a)$$
$$=$$
$$(x_1, ap_1;\ \dots x_n, ap_n; y_1, (1-a)q_1;\ \dots y_n, (1-a)q_n)$$

Expected Value

When you face a lottery, you could start computing the expected value

The expected value of prospect $\boldsymbol{r} = (x_1, p_1;\ \dots x_n, p_n)$ is

$$E(\boldsymbol{r}) = \sum_i p_i \cdot x_i$$

Example

$$\boldsymbol{r} = (1000, 0.25; 500, 0.75)$$

$$E(\boldsymbol{r}) = 0.25 \cdot 1000 + 0.75 \cdot 500$$

A ***fair*** lottery is the lottery that as an expected value equal to zero

The use of the expected value to evaluate the desirability of a lottery can be problematic and you can get wrong conclusions.

St. Petersburg paradox

- A fair coin is tossed repeatedly until a tail appears, ending the game.
- The pot starts at 2 dollars and is doubled every time a head appears.
- Prize is whatever is in the pot after the game ends:
    - 2 dollars if a tail appears on the first toss,
    - 4 dollars if a head appears on the first toss and a tail on the second,
    - 8 dollars if a head appears on the first two tosses and a tail on the third,
    - 16 dollars if a head appears on the first three tosses and a tail on the fourth, etc.
    - 2^k dollars if the coin is tossed *k* times until the first tail appears.

The expected value is ∞:

$$2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} + 8 \cdot \frac{1}{8} + \cdots .. =$$

$$= \sum_{i=1}^{\infty} 2^i \cdot \frac{1}{2^i} =$$

$$= 1 + 1 + 1 + \cdots .. = \infty$$

The experimental evidence is that people are willing to pay only limited amount of money to play this lottery

Solution: the value that people attach to the first dollar of their wealth is larger tat the value they attach to the ith dollar they earn.

A decreasing marginal value can explain this paradox

Questions set 2

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1) Compute the expected value of the following lottery:

$$s = (100,\ 0.30;\ 50,\ 0.50;\ 0,\ 0.2)$$

2) Consider the following compound lottery $q = (r, 0.5;\ \ s = 0.5)$ where $r = (100, 0.4, 50, 0.6)$ and $s = (50, 02; 0, 0.8)$. Write the resulting lottery.

Solution

1) $E(s) = 100 \cdot 0.3 + 50 \cdot 0.5 = 55$

2) $q = (100, 0.5 \cdot 0.4, ; 50, 0.5 \cdot 0.6 + 0.5 \cdot 0.2) = (100, 0.2, ; 50, 0.4)$

Utility over lotteries: Assumptions on preferences over lotteries

To evaluate lotteries we can use an utility function as we do for other economic objects. We need to check if the preferences over lotteries satisfy a number of axioms.

These axioms impose rationality to the individual's behaviour when individuals face choices among lotteries.

a. Completeness

For all lotteries $\boldsymbol{q}$ and $\boldsymbol{r}$ we have that $\boldsymbol{q} \succcurlyeq \boldsymbol{r}\ or\ \boldsymbol{r} \succcurlyeq \boldsymbol{q}$ (or both)

b. Transitivity

For any three lotteries $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$ if $\boldsymbol{q} \succcurlyeq \boldsymbol{r}$ and $\boldsymbol{r} \succcurlyeq \boldsymbol{s}$, then $\boldsymbol{q} \succcurlyeq \boldsymbol{s}$

c. *Continuity*

For any three lotteries $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$ where $\boldsymbol{q} \succcurlyeq \boldsymbol{r}$ and $\boldsymbol{r} \succcurlyeq \boldsymbol{s}$, there exists some probability p such that there is indifference between the middle ranked prospect *r* and the prospect $(\boldsymbol{q}, p; \boldsymbol{s}, 1-p)$, i.e.

$$(\boldsymbol{q}, p; \boldsymbol{s}, 1-p) \sim \boldsymbol{r}$$

Example

$q = (100, 0.5)$, $s = (60, 0.8)$ and $r = (80, 0.6)$

Compute $(\boldsymbol{q}, p; \boldsymbol{s}, 1-p)$. The resulting lottery is:

$$(100, 0.5\, p; 60, 0.8\,(1-p))$$

If the continuity is satisfied then there exists a $\bar{p}$ such that

$(100, 0.5\, \bar{p}; 60, 0.8\,(1-\bar{p})) \sim (80, 0.6)$

$(100, 0.5\, p; 60, 0.8\,(1-p)) \succ (80, 0.6)$ for $p > \bar{p}$

$(100, 0.5\, p; 60, 0.8\,(1-p)) \prec (80, 0.6)$ for $p < \bar{p}$

d. Independence

Any state of the world that results in the same outcome regardless of one's choice can be ignored or cancelled

For any three lotteries $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$ and any $p \in [0, 1]$

if $\boldsymbol{q} \succcurlyeq \boldsymbol{r}$

then $(\boldsymbol{q}, p; \boldsymbol{s}, 1-p) \succcurlyeq (\boldsymbol{r}, p; \boldsymbol{s}, 1-p)$

Example

If $\boldsymbol{q} = (3000)$, $\boldsymbol{r} = (4000, 0.8)$ and $\boldsymbol{q} \succcurlyeq \boldsymbol{r}$

then

$\boldsymbol{q}' = (3000, 0.25)$, $\boldsymbol{r}' = (4000, 0.2)$ and $\boldsymbol{q}' \succcurlyeq \boldsymbol{r}'$

Note that:

Let $\boldsymbol{s}$ be a degenerate prospect that gives you 0 for sure

prospect $\boldsymbol{q}'$ is the compound lottery $\boldsymbol{q}' = (\boldsymbol{q}, 0.25; \boldsymbol{s}, 0.75\ )$ and

prospects $\boldsymbol{r}'$ is the compound lottery $\boldsymbol{r}' = (\boldsymbol{r}, 0.25; \boldsymbol{s}, 0.75\ )$

Utility function over lotteries: Expected Utility

The utility function over lotteries has an *Expected utility form* if for a prospect $\boldsymbol{r} = (x_1, p_1;\ \dots x_n, p_n)$ is given by:

$$U(\boldsymbol{r}) = \sum_i p_i \cdot u(x_i)$$

where $u(x_i)$ is the utility function over outcome x_i

An utility function with expected utility form is called *von Neumann-Morgenstern expected utility function*

Example

$\boldsymbol{r} = (1000, 0.25; 500, 0.75)$ and $u(x_i) = \sqrt{x_i}$

$U(\boldsymbol{r}) = 0.25\sqrt{1000} + 0.75\sqrt{500}$

Representation theorem

Let be X the set of all possible lotteries.

If the preferences over these lotteries are rational (complete and transitive) and satisfy continuity and independence, then there exists a *von Neumann-Morgenstern expected utility function* $U(x)$ such that:

$$q \succcurlyeq r$$

if and only if

$$U(q) \geq U(r)$$

This means that:

If subjects obeys the above axioms, they act choosing the options that maximize *von Neumann-Morgenstern expected utility function*

Questions set 3

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1) Assuming $u = \sqrt{x}$, compute the expected utility of the following lottery:

$$\mathrm{s} = (100,\ 0.30; 64,\ 0.50;\ 0,\ 0.2)$$

2) Assuming $u = \sqrt{x}$, and the lotteries $\mathrm{r} = (100,\ 0.50)$ and s described in the previous question). Which is the preferred lottery?

3) Assume $u = \sqrt{x}$ and consider the following compound lottery $q = (r, p;\ \ s, (1-p))$ where $r = (100, 0.4)$ and $s = (64, 0.2)$. Compute the probability p such that $q \sim t = (81, 0.3)$

1) Assuming $u = \sqrt{x}$, compute the expected utility of the following lottery:

$$\mathrm{s} = (100,\ 0.30; 64,\ 0.50;\ 0,\ 0.2)$$

Solution → $u(s) = 0.3\sqrt{100} + 0.5\sqrt{64} = 7$

2) Assuming $u = \sqrt{x}$, and the lotteries $\mathrm{r} = (100,\ 0.50)$ and s described in the previous question). Which is the preferred lottery?

Solution → $u(s) = 0.5\sqrt{100} = 5$ → lottery s is preferred

3) Assume $u = \sqrt{x}$ and consider the following compound lottery $q = (r, p;\;\; s, (1-p))$ where $r = (100, 0.4)$ and $s = (64, 0.2)$.

Compute the probability p such that $q \sim t = (81, 0.3)$

Solution:

$$q = (100, 0.4 \cdot p, ; \quad 64, 0.2 \cdot (1-p))$$

$$U(q) = 0.4p\sqrt{100} + 0.2(1-p)\sqrt{64} = 4p + 1.6(1-p) = 1.6 + 2.4p$$

$$U(t) = 0.3 \cdot \sqrt{81} = 2.7$$

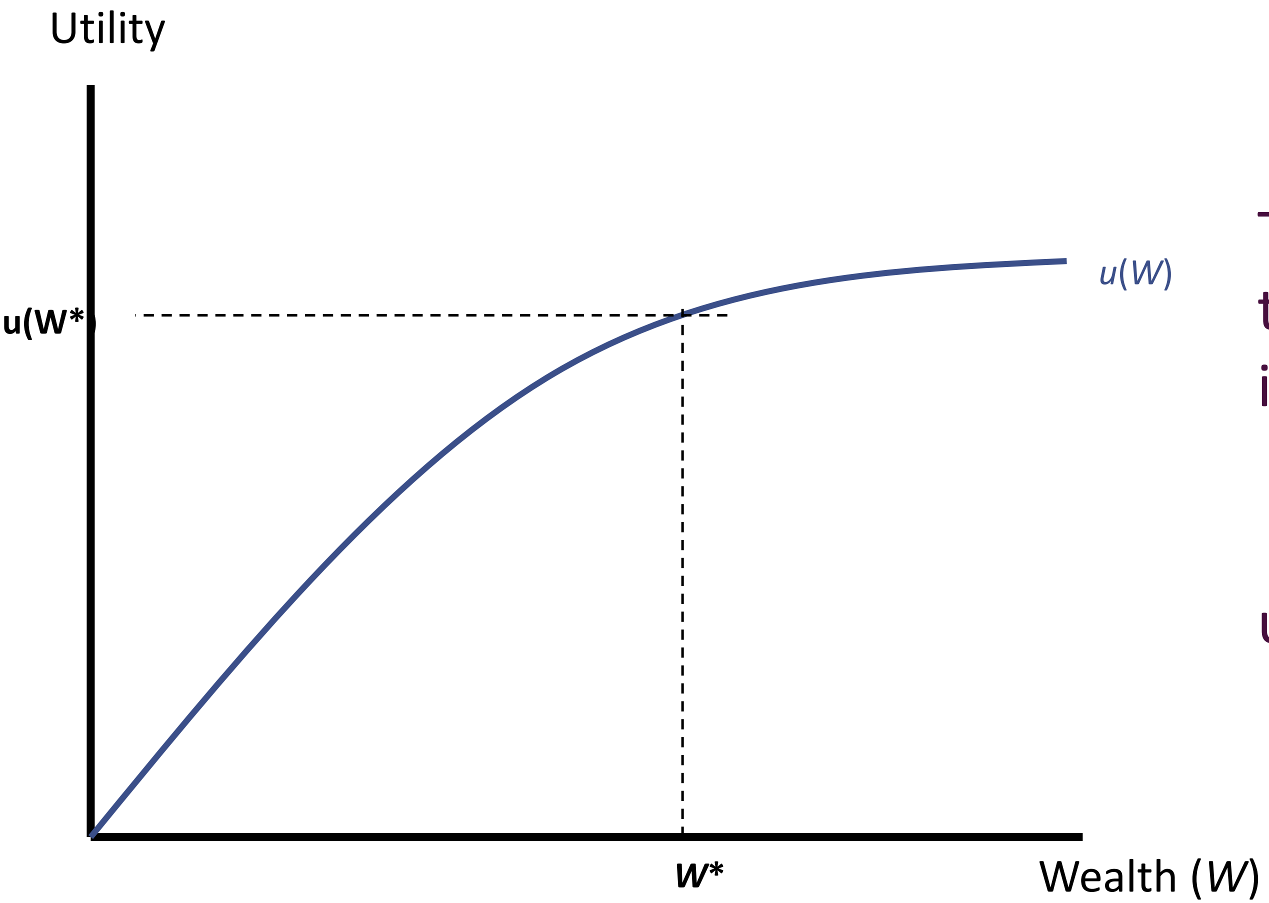
we need to satisfy the condition

$$U(t) = U(q)$$

$$2.7 = 1.6 + 2.4p$$

$$p = \frac{1.1}{2.4} = 0.46$$

Risk Aversion



The curve is concave to reflect the assumption that marginal utility diminishes as wealth increases

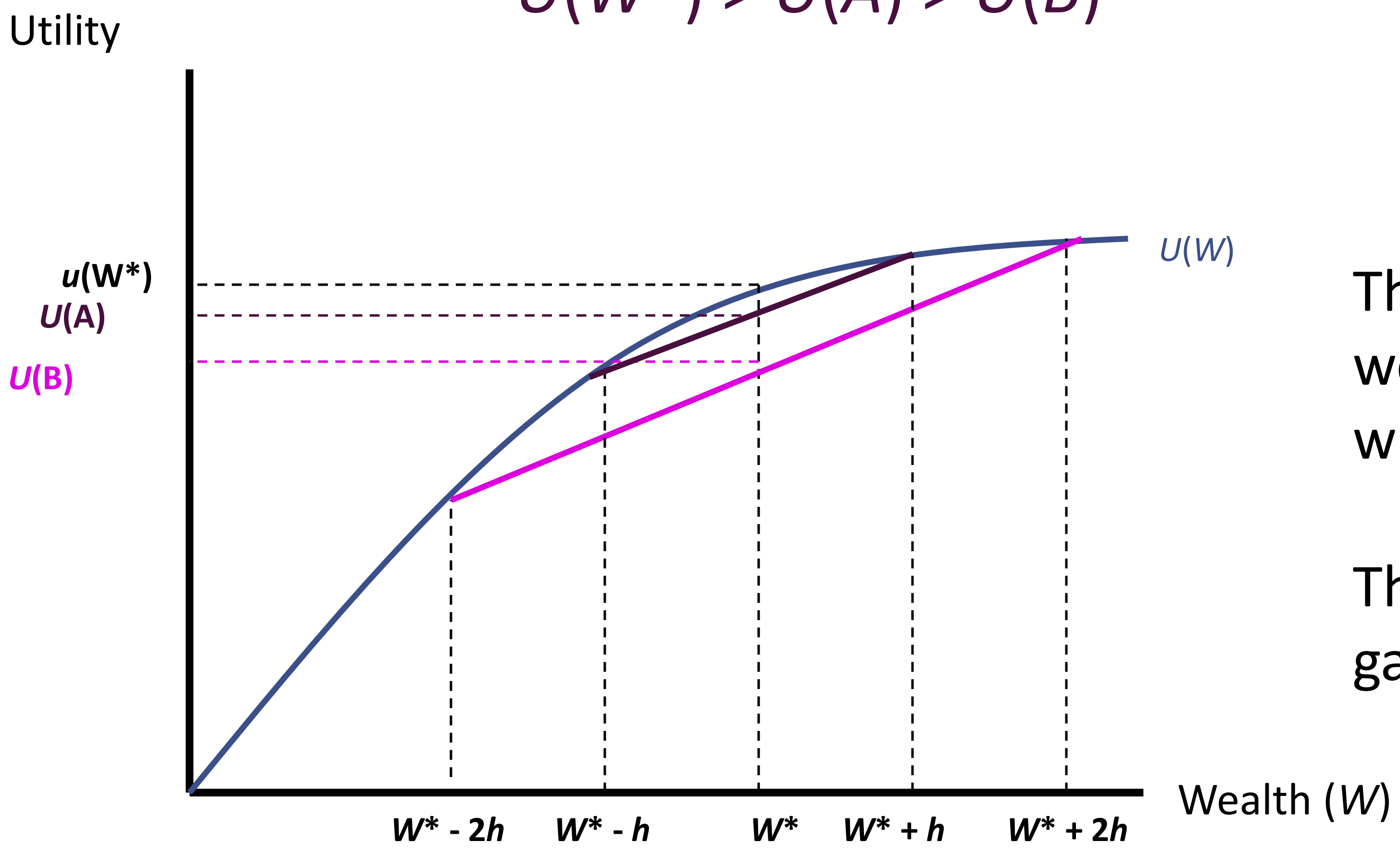
u(W*) is the individual's current level of utility

Suppose that the person is offered two fair gambles:

A) a 50-50 chance of winning or losing $\$h$ → $U(A) = ½\, u(W^* + h) + ½\, u(W^* - h)$

B) a 50-50 chance of winning or losing $\$2h$ → $U(B) = ½\, u(W^* + 2h) + ½\, u(W^* - 2h)$

$$U(W^*) > U(A) > U(B)$$



The person will prefer current wealth to that wealth combined with a fair gamble

The person will also prefer a small gamble over a large one

Risk Attitudes

From the previous example we learned that if the $u(x)$ is concave the subject is risk averse. In general individuals could also be risk neutral or risk lovers. So we need some more precise definition.

A decision maker is ***risk neutral*** if he is indifferent between receiving a lottery's expected value and playing the lottery.

Consider $\boldsymbol{r} = (x_1, p_1;\ \ldots x_n, p_n)$ then:

$$u\left(\sum_i p_i \cdot x_i\right) = \sum_i p_i \cdot u(x_i)$$

A decision maker is risk neutral if its utility function is linear, i.e. $\boldsymbol{u(x) = a + b\,x}$

linear utility function → constant marginal utility

Example:

Lottery $r = (10, 0.6; 2, 0.4)$

Its expected value is $E(r) = 10 \cdot 0.6 + 2 \cdot 0.4 = 6.8$

Risk neutral agent is indifferent between receiving (and playing) lottery r and receiving 6.8 for sure:

$$u(6.8) = 0.6 \cdot u(10) + 0.4 \cdot u(2)$$

A decision maker is ***risk averse*** if he prefers receiving the lottery's expected value instead of playing the lottery.

Consider $\boldsymbol{r} = (x_1, p_1;\ \ldots x_n, p_n)$ then:

$$u\left(\sum_i p_i \cdot x_i\right) > \sum_i p_i \cdot u(x_i)$$

A decision maker is risk averse if its utility function is strictly concave, i.e. $\boldsymbol{u''(x) < 0}$

Strictly concave utility function → decreasing marginal utility

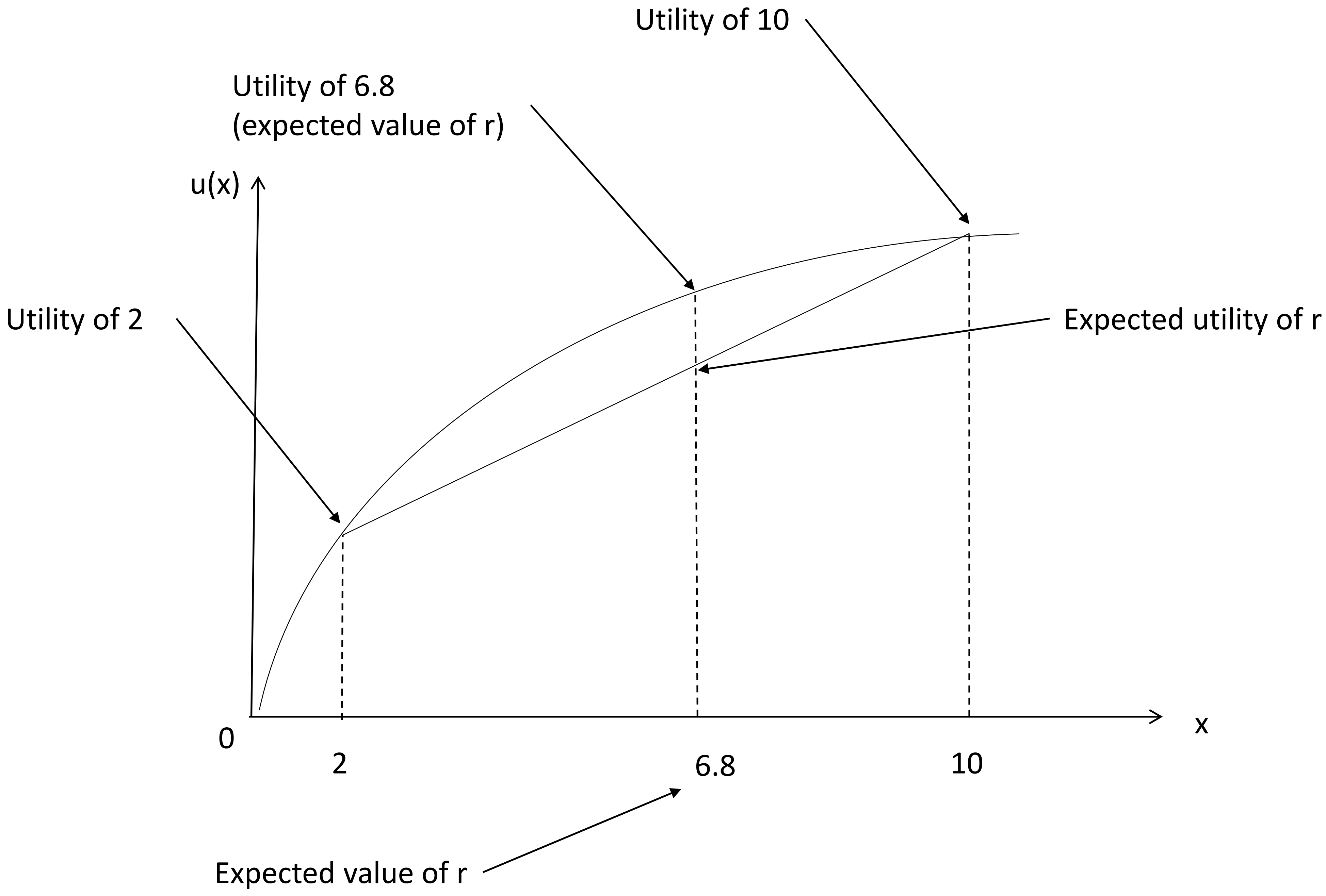
Example:

Lottery $r = (10, 0.6; 2, 0.4)$

Its expected value is $E(r) = 10 \cdot 0.6 + 2 \cdot 0.4 = 6.8$

Risk averse agent prefers receiving 6.8 for sure that receiving (and playing) lottery r

$$u(6.8) > 0.6 \cdot u(10) + 0.4 \cdot u(4)$$


Utility of 10
Utility of 6.8
(expected value of r)
u(x)
Utility of 2
Expected utility of r
x
0
2
6.8
10
Expected value of r

A decision maker is ***risk seeking (or risk lover)*** if he prefers playing the lottery instead of receiving its expected value.

Consider $\boldsymbol{r} = (x_1, p_1;\ \ldots x_n, p_n)$ then:

$$u\left(\sum_i p_i \cdot x_i\right) < \sum_i p_i \cdot u(x_i)$$

A decision maker is risk seeking if its utility function is strictly convex, i.e. $\boldsymbol{u''(x) > 0}$

Strictly convex utility function → increasing marginal utility

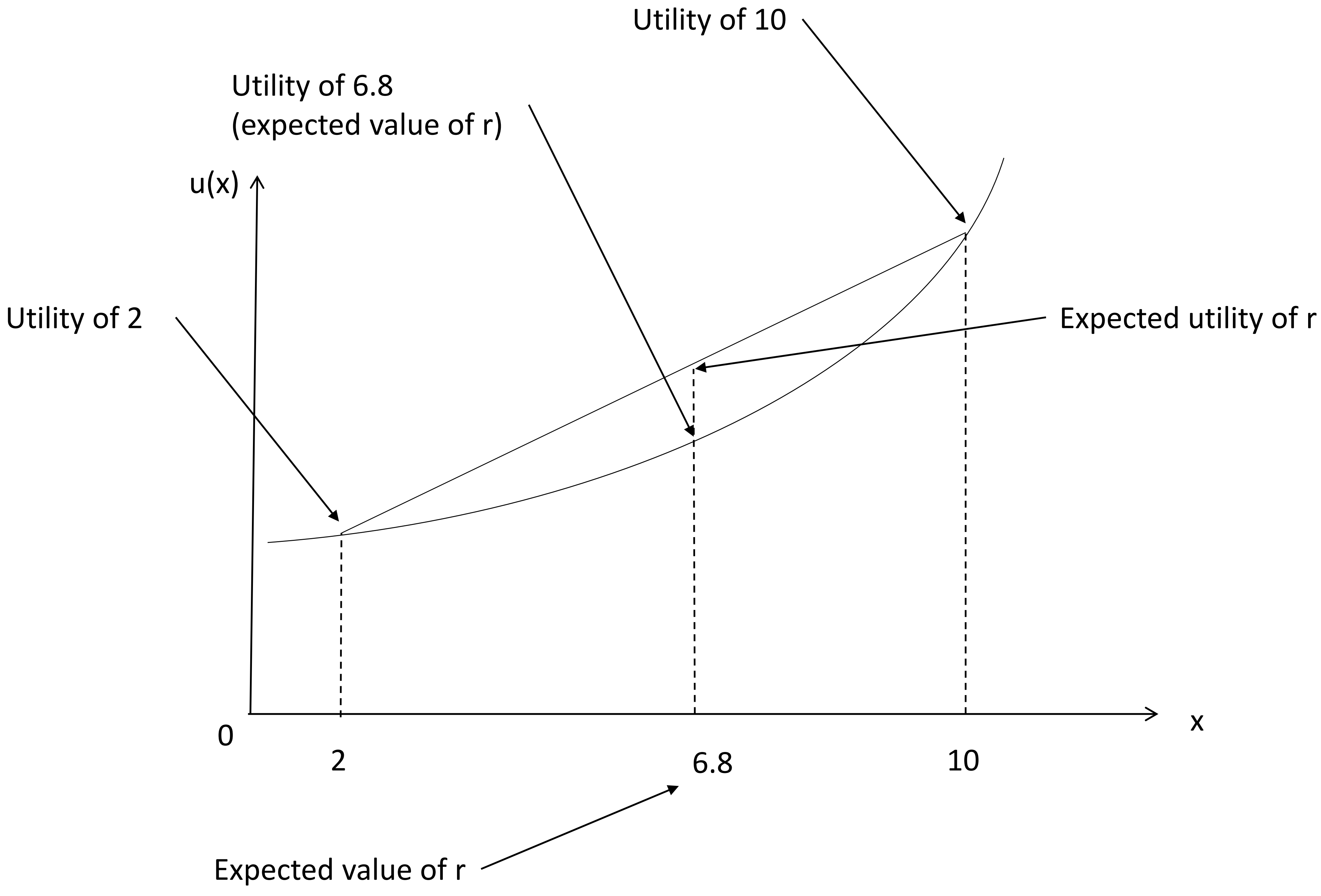
Example:

Lottery $r = (10, 0.6; 2, 0.4)$

Its expected value is $E(r) = 10 \cdot 0.6 + 2 \cdot 0.4 = 6.8$

Risk lover agent prefers receiving (and playing) lottery r that receiving 6.8 for sure

$$u(6.8) < 0.6 \cdot u(10) + 0.4 \cdot u(4)$$



Utility of 10
Utility of 6.8
(expected value of r)
u(x)
Utility of 2
Expected utility of r
0
x
2
6.8
10
Expected value of r

All these results are proved by Jensen’s Inequality

Let x be a random variable where $E(x)$ is its expected value and $u(x)$ is a concave function then:

$$u\big(E(x)\big) \geq E\big(u(x)\big)$$

$u(x)$ is a convex function then:

$$f\big(u(x)\big) \leq E\big(u(x)\big)$$

Questions set 4
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Let be $x > 0$

1) If the subject preferences are represented by $u = \sqrt{x}$, we can say that this subject is risk….
2) If the subject preferences are represented by $u = \ln x$, we can say that this subject is risk….
3) If the subject preferences are represented by $u = x^a$ $a > 1$, we can say that this subject is risk….

solution

Let be $x > 0$

1) If the subject preferences are represented by $u = \sqrt{x}$, we can say that this subject is risk…. **Averse**

$$\mathrm{u}' = \frac{1}{2}x^{-\frac{1}{2}} \longrightarrow \mathrm{u}'' = -\frac{1}{4}x^{-\frac{3}{2}} < 0$$

1) If the subject preferences are represented by $u = \ln x$, we can say that this subject is risk…. **Averse**

$$\mathrm{u}' = x^{-1} \rightarrow \mathrm{u}'' = -x^{-2} < 0$$

1) If the subject preferences are represented by $u = x^{a}\ a > 1$, we can say that this subject is risk…. **Lover**

$$\mathrm{u}' = ax^{a-1} \rightarrow \mathrm{u}'' = \mathrm{a}\,(\mathrm{a}-1)x^{a-2} > 0$$