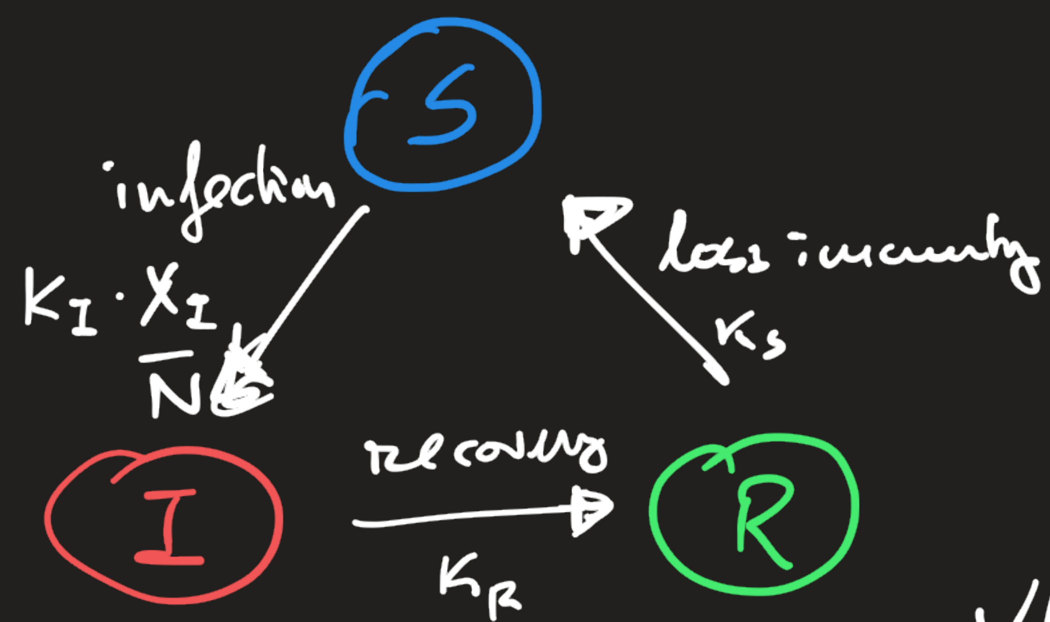
POPULATION CTMCs

X_I: # infected individuals



$Y_j(t)$: state of j^{th} agent at time t.

$Y(t) = (Y_1(t), \ldots, Y_N(t))$ N agents.

$S = \{(\sigma_1, \ldots, \sigma_N), \sigma_j \in \{S, I, R\}\}, |S| = 3^N$

$X_\alpha(t) = \sum_{j=1}^{N} \mathbb{I}(\sigma_j = \alpha)$ $\alpha \in \{S, I, R\}$

= # agents in state α at time t.

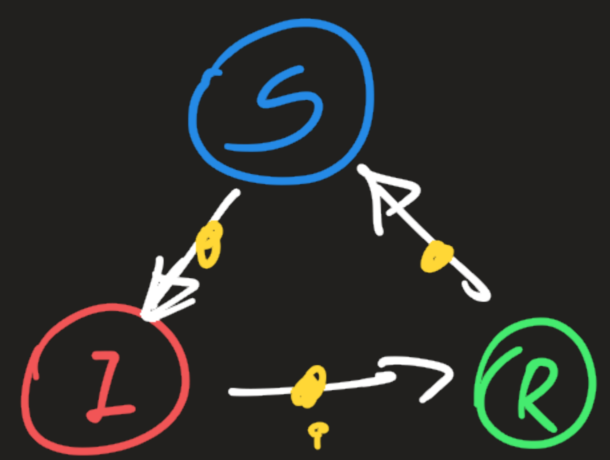
POPULATION BASED DESCRIPTION

$|S| = O(N^2)$ (in general $|S| = O(N^K)$, K = # of agent states.

$Y(t)$ and $X(t) = (X_S(t), X_I(t), X_R(t))$ are STOCHASTICALLY EQUIVALENT.

CHEMICAL REACTION NETWORKS

"POPULATION MODELS!"



$$\underline{I} \xrightarrow[\underbrace{K_R \cdot X_I}]{\text{recovery}} R$$

$$\underline{R} \xrightarrow[\underbrace{K_S \cdot X_R}]{\text{imm. loss}} S$$

$$\underbrace{S + I}_{\text{REACTANTS}} \xrightarrow[\underbrace{\frac{K_I \cdot X_I \cdot X_S}{N}}_{\text{Reactions / Transitions}}]{\text{infection}} \underbrace{I + I}_{\text{PRODUCTS}}$$

Formally, a population model is specified by $(S, X, \mathcal{R}, x_0)$

- X = POP. VARS. $X = (X_1, \ldots, X_n)$ or (X_S, X_I, X_R) ...
  SPECIES
- S is the state space $S \subseteq \mathbb{N}^n$
- x_0 INITIAL STATE
- $\mathcal{R}$ REACTIONS, $\rho \in \mathcal{R}$ is $\rho: r_1 X_1 + \ldots + r_n X_n \longrightarrow p_1 X_1 + \ldots + p_n X_n$
  $r_i \in \mathbb{N}$ $\quad$ $p_i \in \mathbb{N}$
  $f_\rho(X) \in \mathbb{R}_{\geq 0}$ is the RATE FUNCTION

(X, S, R, x_0) : $X = (X_1, \dots, X_n)$, n species

$S \subseteq \mathbb{N}^n$ state space

$x_0 \in S$ initial state

R : $(r_1 X_1 + \dots + r_n X_n \longrightarrow p_1 X_1 + \dots + p_n X_n \; , f(x))$ reactions

Example

$$\begin{array}{llll} S + I \to I + I & r = (1,1,0);\ p = (0,2,0) & ,\ v = (-1,1,0) & p - r = v \text{ update} \\ I \to R & r = (0,1,0) & p = (0,0,1) & - v = (0,-1,1) \\ R \to S & r = (0,0,1) & p = (1,0,0) & - v = (1,0,-1) \end{array}$$

$$\Sigma : \begin{array}{c} \\ S \\ I \\ R \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \end{array}$$ stoichiometric matrix

- if $\Sigma y = 0$, then y a t-invariant (integer solutions)
  $|y| = |R|$ $\qquad y = (1,1,1)^T$
- if $\Sigma^T \cdot z = 0$, then z is a p-invariant : $\boxed{z^T \cdot X}$ is a constant for the dynamics
  $|z| = n$ $\qquad$ species invariant $\qquad$ conservation law

$z = (1,1,1)^T \qquad z^T X : X_S + X_I + X_R \equiv \text{const}$

$X_S(t) + X_I(t) + X_R(t) = N$

MASTER EQUATION

$\Updownarrow$

KOLMOGOROV EQUATION

$x, x_0 \in S$

$$P(x,t) = P(X(t) = x \mid X(0) = x_0)$$

$$\boxed{\frac{dP(x,t)}{dt} = \sum_{\rho_j \in R} f_j(x - v_j) P(x - v_j, t) - \sum_{\rho_j \in R} f_j(x) P(x,t)}$$

one equation for each state in S.

$R_j(t) \equiv$ COUNT OF HOW MANY TIMES ρ_j FIRED UPTO TIME t.

$$\boxed{X(t) = X(0) + \sum_{j \in R} v_j \cdot R_j(t)}$$

R_j and R_i, $i \neq j$ are independent, R_j is a time inhomogeneous Poisson with rate $f_j(X(s))$, cum rate is

$$\boxed{\Lambda_j(t) = \int_0^t f_j(X(s))\,ds}$$

$$X(t) = X(0) + \sum_{j \in R} v_j \cdot \mathrm{Poisson}(\Lambda_j(t))$$

CTMC SIMULATION

- DIRECT SIMULATION METHOD

SAMPLE_NEXT.STATE (x, t)

FOR ρ IN REACTIONS :

$\lambda_\rho = f_\rho(x)$

$t_\rho \sim \mathrm{EXP}(\lambda_\rho)$

LET $\Delta t, j =$ min t_ρ, argmin t_ρ

RETURN $x + v_j, t + \Delta t$

SIMULATION (T, x_0)

$x = x_0$

$t = 0$

WHILE $t < T$:

$x, t \leftarrow$ SIMULATE_NEXT_STATE (x,t)

...

COMPLEXITY in Terms of CALLS TO PRNG!

$K = |R|$

$O(K)$ PER STEP

SSA (DOOB-GILLESPIE)

SIMULATION_STEP(x, t)

FOR j IN REACTIONS:

$\rho_j = f_j(x)$

$\rho = \sum_j \rho_j$

$\Delta t \sim \mathrm{EXP}(\rho)$

$j \sim \mathrm{CATEGORICAL}\left(\frac{\rho_1}{\rho}, \dots, \frac{\rho_K}{\rho}\right)$ ← BINARY TREE WITH PARTIAL SUMS ON INNER NODES

$O(K)$ to $O(\log K)$

$x = x + v_j$

$t = t + \Delta t$

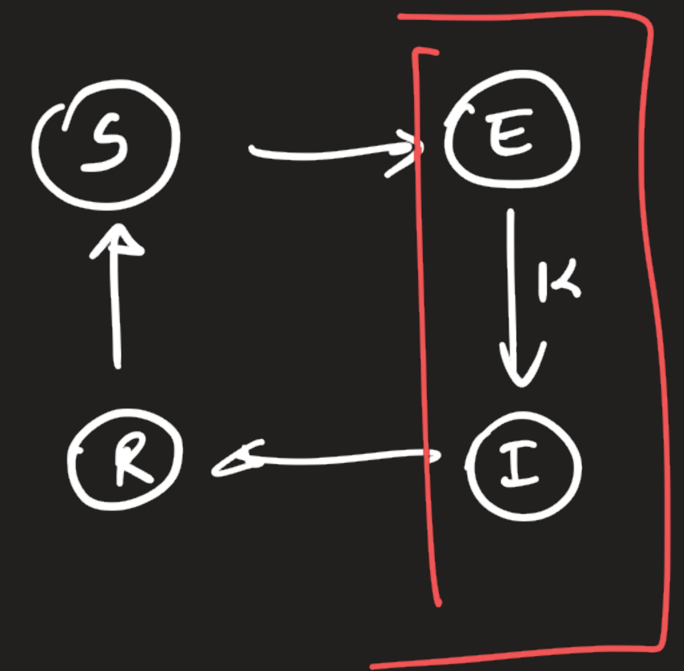
RETURN x, t

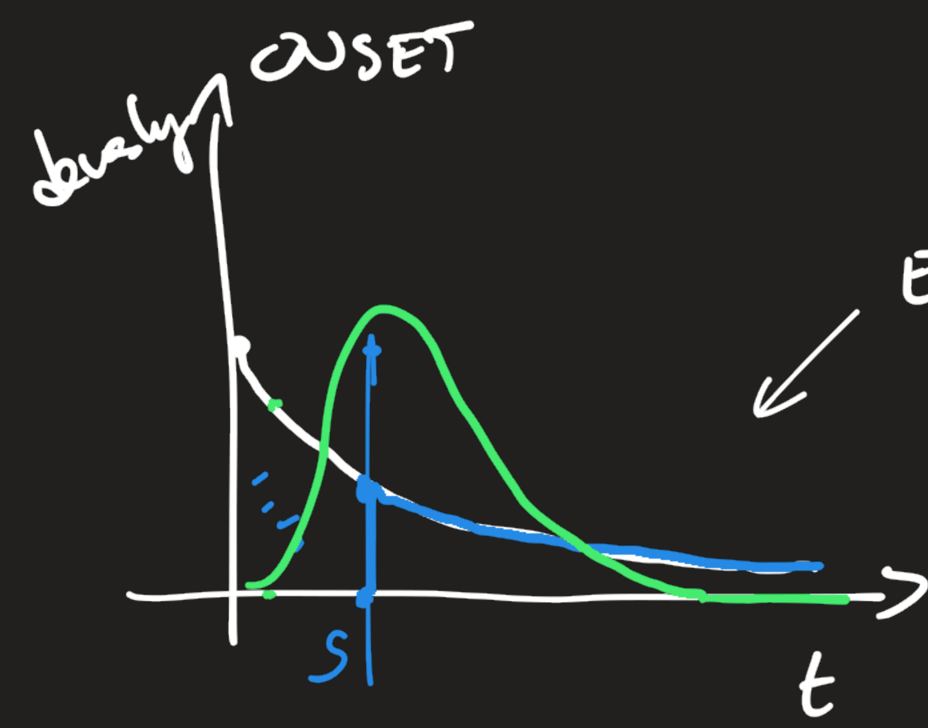
DEPENDENCY GRAPH (DIGRAPH)

$i \to j$ IF $f_j(x)$ DEPENDS EXPLICITLY ON x_i

$j \to i$ IF REACTION j MODIFIES x_i.

→ COMPLEXITY $O(1)$ PRNG per step (exactly 2)



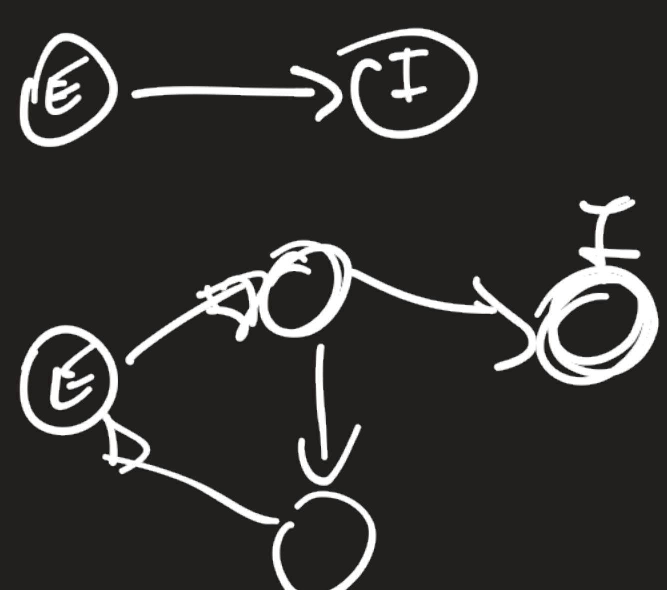


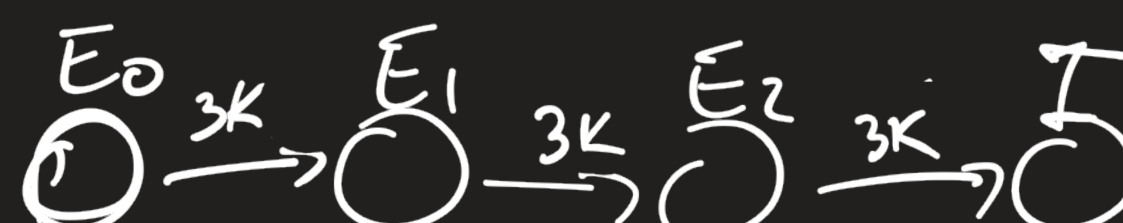
EXP(K)

$p(x) = Ke^{-Kt}$

PHASE-TYPE distribution: HITTING TIME OF S IN A CTMC with S absorbing state.

dense in distributions over $\mathbb{R}_{\geq 0}$





ERLANG DISTRIBUTION

DISCRETE-EVENT SIMULATION

Idea

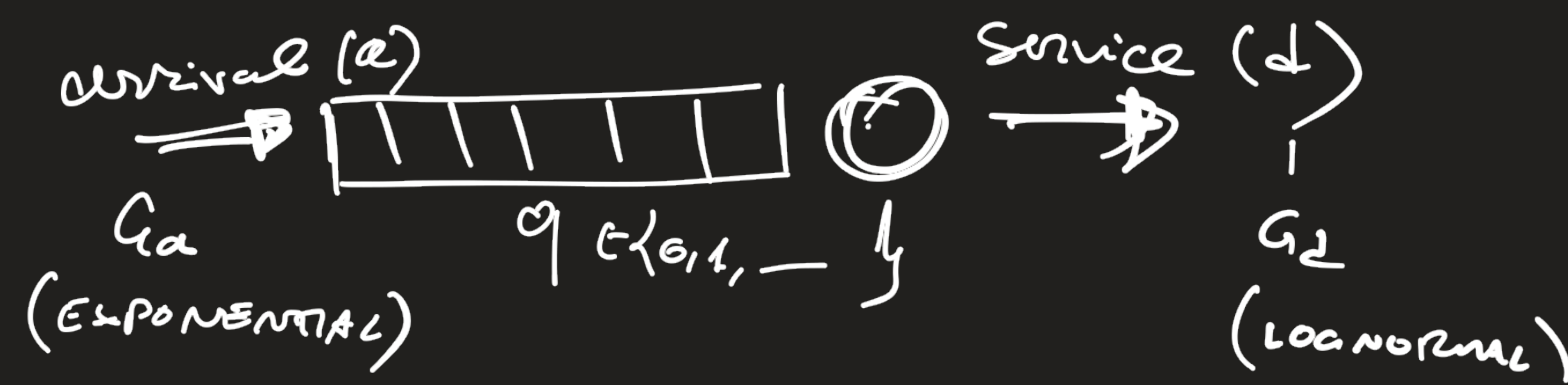
e: $t=0$ TIME TO FIRE $t_e \sim \underbrace{G_e}$ generic distribution

$\text{EXP}(\lambda)$ $\left(t = \frac{1}{\lambda}\xi, \ \xi \sim \text{EXP}(1) = -\log U, \ U \sim \text{UNIF}(0,1) \right)$

$t = t_1 > 0$ TIME ELAPSED: e WILL FIRE IN $t_e - t_1$ UNITS OF TIME

- STATE SPACE S, countable, $s \in S$.
- SET OF EVENTS E, $e \in E$ IS AN EVENTS
- SET OF ENABLED EVENTS $\forall s \in S$, $\Gamma(s) \subseteq E$ enabled events
- $\forall e \in E$, we associate $T_e \in \mathbb{R}_{\geq 0}$, random, $T_e \sim \underbrace{G_e}_{\text{general time distribution}}$
- JUMP KERNEL $p(s' \mid s, e)$: prob of jumping from s to s' given e

$$\sum_{s'} p(s' \mid s, e) = 1$$



$S = \{0, 1, 2, \dots\} = \mathbb{N}$ state space of M/G/1 queue

$E = \{a, d\}$ $\qquad$ $\Gamma(0) = \{a\}$, $\Gamma(q) = \{a, d\}$, $q > 0$

$p(q+1 \mid q, a) = 1$

$p(q-1 \mid q, d) = 1$, $q > 0$ $\qquad$ $G_a = EXP(\lambda)$

$G_d = \text{LOGNORMAL}(\mu, \sigma^2)$ ✓

$r : S \times E \to \mathbb{R}_{\geq 0}$, $\quad$ $e \notin \Gamma(s)$ iff $r(s, e) = 0$ $\quad$ GENERALIZED SEMI-MARKOV PROCESS

$c_e \sim G_e \qquad t_e = \dfrac{c_e}{r(s, e)} \qquad t : \; c_e' = c_e - t \cdot r(s, e)$

CURRENT SIMULATION TIME : $t \in \mathbb{R}_{\geq 0}$

CURRENT STATE : $s \in S$

SCHEDULING OF EVENTS : $f = [\underbrace{(e_0, c_0), (e_1, c_1), \ldots, (e_k, c_k)}_{\text{list of ENABLED EVENTS + THEIR CLOCK i.e. TIME TO FIRE.}}]$

INITIALIZATION PHASE

$t = 0$
$s = s_0$
$f = \emptyset$
FOR EACH $e \in \Gamma(s)$:
    $c_e = \text{SAMPLE}(G_e)$
    ADD (e, c_e) TO f

WHILE (not Termination)
    $t, s, f \leftarrow \text{SIM\_STEP}(t, s, f)$
    "store code"

SIM_STEP(t, s, f)

$\Delta t = \min_{t_e} \{ t_e = \frac{c_e}{r(s,e)} \mid e \in \Gamma(s) \}$ $\quad e \in f|_e$

$e^* = \arg\min \{ t_e = \frac{c_e}{r(s,e)}, e \in \Gamma(s) \}$

$s^* \sim p(s' \mid s, e^*)$

$t = t + \Delta t$

UPDATE$(f, s, s^*, \Delta t, e^*)$

$s = s^*$

RETURN t, s, f

$(e, t_e = \frac{c_e}{r(s,e)})$

$\downarrow$

$\boxed{(t_e - \Delta t) \frac{r(s,e)}{r(s^*,e)}}$

update if we use a PRIORITY QUEUE

UPDATE$(f, s, s^*, \Delta t, e^*)$:

REMOVE(e^*, c_e)

FOR EACH $(e, c_e) \in f$:

IF $e \notin \Gamma(s^*)$, remove(e, c_e)

ELSE $c_e = c_e - \Delta t \cdot r(s,e) > 0$

FOR $e \in \Gamma(s^*)$ AND $e \notin f_e$

ADD$(f, (e, c_e))$, $c_e \sim G_e$