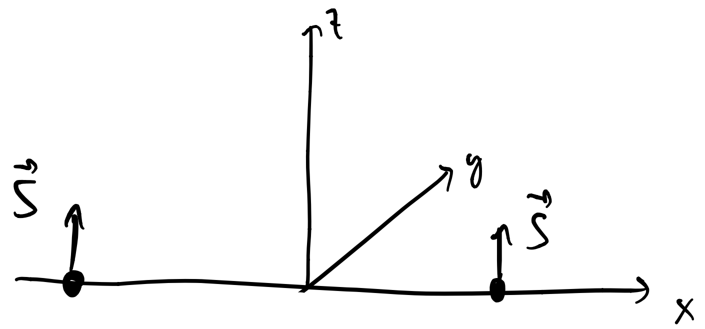
INTEGRALE SUI CAMMINI PER FERMIONI

[S.14.6]
[PS.9.5]

Abbiamo visto che sotto rotazioni di 2π

$$\psi(x) \xrightarrow{\vartheta_z = 2\pi} -\psi(x).$$

Consideriamo 2 fermioni identici posizionati a $C\hat{x}$ e $-C\hat{x}$, con spin allineati a $\hat{z}$



lo stato è:

$$|\psi_{12}\rangle = |\psi_{\uparrow}(\vec{x})\,\psi_{\uparrow}(-\vec{x})\rangle$$

Ruotiamo di π attorno a $\hat{z}$: ogni spinore ha: $\Lambda_s(\pi) = i$

$$|\psi_{12}\rangle \xrightarrow{\vartheta_z = \pi} -|\psi_{21}\rangle \underset{\uparrow}{=} -|\psi_{12}\rangle$$

perché i due fermioni sono identici.

$\Rightarrow$ la funzione d'onda cambia segno sotto scambio di due spinori.

I fermioni **anticommutano**.

A livello CLASSICO: $\{\psi(x), \chi(y)\} = 0$

Introduciamo il concetto di numeri che anticommutano.

ALGEBRA DI GRASSMAN: insieme $\mathcal{G}$ di oggetti generati da una base $\{\vartheta_i\}$ NUMERI DI GRASSMAN

$\{\vartheta_i\}$ soddisfano:
$$\begin{cases} \vartheta_i \vartheta_j = -\vartheta_j \vartheta_i \\ \vartheta_i + \vartheta_j = \vartheta_j + \vartheta_i \\ \alpha\vartheta \in \mathcal{G} \quad \text{dato} \quad \vartheta \in \mathcal{G} \text{ e } \alpha \in \mathbb{C} \\ \exists\, 0 : \quad \vartheta_i + 0 = \vartheta_i \end{cases}$$

$\Rightarrow \quad \vartheta_i^2 = 0 \quad : \quad \vartheta_i^2 = -\vartheta_i^2 \rightarrow 2\vartheta_i^2 = 0$

- Per $\boxed{\text{un } \vartheta}$, l'elemento generico dell'algebra è
$$g = a + b\vartheta \qquad a, b \in \mathbb{C}$$
- Per due $\{\vartheta_1, \vartheta_2\}$: $\quad g = a + b\vartheta_1 + c\vartheta_2 + f\vartheta_1\vartheta_2$
- $(\vartheta_i\vartheta_j)$ è "bosonico" : $\quad \vartheta_i\vartheta_j\vartheta_a = \vartheta_a\vartheta_i\vartheta_j \qquad$ (o ogni 2n)
- $(\vartheta_i \dots \vartheta_{2n+1})$ è "fermionico" : $\quad \vartheta_i\vartheta_j \dots \vartheta_{2n+1}\vartheta_a = -\vartheta_a\vartheta_i \dots \vartheta_{2n+1}$

INTEGRALE $\int_{-\infty}^{+\infty} dx\, f(x) \longleftrightarrow \int d\theta\, f(\theta)$

È una mappa da $\mathcal{G} \to \mathbb{C}$

- linearità: $\int d\theta_1 \dots d\theta_n (sX + tY) = s\int d\theta_1 \dots d\theta_n X + t\int d\theta_1 \dots d\theta_n Y$

  $s,t \in \mathbb{C}$; $X, Y \in \mathcal{G}$

- $d\theta$ anticommuta come θ, anche $\int d\theta$ $\Rightarrow$ $\int d\theta = 0$

  $\int d\theta (a + b\theta) \equiv a\int d\theta + b\int d\theta\, \theta = b$ DEF: $\boxed{\int d\theta\, \theta = 1}$

  $\frac{d}{d\theta}(a + b\theta) \equiv b$

  $\Rightarrow \frac{d}{d\theta} = \int d\theta$ per numeri Grassmaniani

- Per più θ_i:

  $\int d\theta_1 \dots d\theta_n X \equiv \frac{\partial}{\partial\theta_1} \dots \frac{\partial}{\partial\theta_n} X \Rightarrow \int d\theta_1 \dots d\theta_n\, \theta_n \dots \theta_1 = 1$

  NOTA: $\int d\theta_1 d\theta_2\, \theta_2 \theta_1 = -\int d\theta_1 d\theta_2\, \theta_1 \theta_2 = 1$

- Cambio di variabili

  $\int_{-\infty}^{+\infty} dx\, f(x) = \int_{-\infty}^{+\infty} dx\, f(x+a)$ | $\int d\theta (A + B\theta) = \int d\theta (A + B(\theta + X))$

  $a: \frac{d}{dx} a = 0$ | $X: \frac{\partial}{\partial\theta} X = 0$

- $\int d\theta_1 d\theta_2\, e^{-\theta_1 A_{12} \theta_2} = \int d\theta_1 d\theta_2 \left(1 - \theta_1 A_{12} \theta_2\right) = A_{12}$

termini sup hanno $\theta_1^2 = 0$ o $\theta_2^2 = 0$.

- Numeri complessi:

$\theta = \dfrac{\theta_1 + i\theta_2}{\sqrt{2}}$, $\bar{\theta} \equiv \theta^* = \dfrac{\theta_1 - i\theta_2}{\sqrt{2}}$ $\to$ $\int d\bar{\theta} d\theta\, \theta\bar{\theta} = 1$

$(\theta\eta)^* = \bar{\eta}\,\bar{\theta} = -\bar{\theta}\,\bar{\eta}$ (come prendere l'Hermitiano di operatori)

- Prendiamo n θ_i e n $\bar{\theta}_i$ (tutte variabili di Grassman indipendenti)

$$\int d\bar{\theta}_1 \dots d\bar{\theta}_2\, d\theta_1 \dots d\theta_n\, e^{-\bar{\theta}_i A_{ij} \theta_j} =$$

$$= \int d\bar{\theta}_1 \dots d\bar{\theta}_2\, d\theta_1 \dots d\theta_n \left(1 - \bar{\theta}_i A_{ij} \theta_j + \frac{1}{2} (\bar{\theta}_i A_{ij} \theta_j)(\bar{\theta}_k A_{kl} \theta_l) + \dots \right)$$

L'unico termine non nullo è quello con tutti gli θ_i e $\bar{\theta}_i$

$$= \frac{1}{n!} \sum_{\text{perm}\{i_n\}} \pm A_{i_1 i_2} \dots A_{i_{n-1} i_n} = \det(A)$$

$$\boxed{\begin{aligned} &\int d\bar{\theta}_1 \dots d\bar{\theta}_2\, d\theta_1 \dots d\theta_n\, e^{-\bar{\theta}_i A_{ij} \theta_j} = \det(A) \\ &\int dx_1 \dots dx_n\, e^{-\frac{1}{2} x_i A_{ij} x_j} = \sqrt{\frac{(2\pi)^n}{\det(A)}} \end{aligned}}$$

- Prendiamo altre $2n$ variabili di Grassman η_i e $\bar{\eta}_i$:

$$\int d\bar{\theta}_1 \dots d\bar{\theta}_2 d\theta_1 \dots d\theta_n \, e^{-\bar{\theta}_i A_{ij} \theta_j + \bar{\eta}_i \theta_i + \bar{\theta}_i \eta_i} =$$

$$= e^{\bar{\eta}_i A_{ij} \eta_j} \int d\bar{\theta}_1 \dots d\bar{\theta}_2 d\theta_1 \dots d\theta_n \, e^{-(\bar{\theta}_i - \bar{\eta}_i A^{-1}) A (\theta_i - A^{-1} \eta_i)} = \det(A) \, e^{\bar{\eta}_i A^{-1}_{ij} \eta_j}$$

Limite del continuo

Prendiamo il limite del continuo:

$$i \to x \qquad \theta_i \to \psi(x)$$

$$\bar{\theta}_i \to \bar{\psi}(x)$$

$$\psi(x) = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_p}} \left(\underbrace{a_s(p)}_{} u_s(p) e^{-ipx} + \underbrace{b^\dagger_s(p)}_{} v_s(p) e^{ipx} \right)$$

Variabili di Grassman

Per definire il funzionale generatore introduciamo due correnti esterne, $\eta(x)$ e $\bar{\eta}(x)$, anch'esse Grassmaniane:

$$Z[\bar{\eta}, \eta] = \int D[\bar{\psi}(x)] \, D[\psi(x)] \, \exp\left\{ i \int d^4 x \left[\bar{\psi} (i\partial\!\!\!/ - m + i\varepsilon) \psi + \bar{\eta} \psi + \bar{\psi} \eta \right] \right\}$$

$$= Z[0,0] \, e^{-\int d^4 x \, d^4 y \, \bar{\eta}(x) \, S_F(x-y) \, \eta(y)} ,$$

dove: $Z[0,0] = \det(i\partial\!\!\!/ - m)$

Abbiamo anche definito: $(i\not{\partial}_x - m + i\varepsilon) S_F(x-y) = i\delta^4(x-y)$

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{\not{p} - m + i\varepsilon} e^{-ip(x-y)}$$

$$(i\not{\partial}_x - m + i\varepsilon) S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{\cancel{(\not{p} - m + i\varepsilon)}} \cancel{(\not{p} - m + i\varepsilon)} e^{-ip(x-y)} = i\delta^4(x-y) \quad \checkmark$$

Nota: $$\frac{\delta}{\delta\eta(x)} \bar{\vartheta}(x)\eta(x) = -\frac{\delta}{\delta\eta(x)} \eta(x)\bar{\vartheta}(x) = -\bar{\vartheta}(x)$$

La funzione a due punti è data da:

$$\langle 0 | T\{\psi_\alpha(x)\bar{\psi}_\beta(y)\} | 0 \rangle = Z[0,0]^{-1} \left(-i\frac{\delta}{\delta\bar{\eta}_\alpha(x)}\right)\left(+i\frac{\delta}{\delta\eta_\beta(y)}\right) Z[\bar{\eta},\eta]\bigg|_{\bar{\eta}=\eta=0} =$$

indici spinoriali

$$= Z[0]^{-1}\int D\bar{\psi}D\psi \left(-i\frac{\delta}{\delta\bar{\eta}_\alpha(x)}\right) \bar{\psi}_\beta(y)\, e^{i\langle\bar{\psi}(i\not{\partial}-m+i\varepsilon)\psi + \bar{\eta}\psi + \bar{\psi}\eta\rangle} =$$

$$= Z[0]^{-1}\int D\bar{\psi}D\psi\, \bar{\psi}_\beta(y)(-\psi_\alpha(x))\, e^{i\langle\bar{\psi}(i\not{\partial}-m+i\varepsilon)\psi + \bar{\eta}\psi + \bar{\psi}\eta\rangle} =$$

$$= Z[0]^{-1}\int D\bar{\psi}D\psi \left(\psi_\alpha(x)\bar{\psi}_\beta(y)\right) e^{i\langle\bar{\psi}(i\not{\partial}-m+i\varepsilon)\psi + \bar{\eta}\psi + \bar{\psi}\eta\rangle} =$$

$$= \frac{\partial^2}{\partial\bar{\eta}_\alpha(x)\,\partial\eta_\beta(y)}\left(1 - \langle\bar{\eta}\, S_F\, \eta\rangle\right) = \frac{\partial}{\partial\bar{\eta}_\alpha(x)}\int d^4w\, \bar{\eta}_\gamma(w) S_F^{\gamma\beta}(w-y) = S_F^{\alpha\beta}(x-y)$$

$$= \int \frac{d^4p}{(2\pi)^4}\left[\frac{i}{\not{p}-m+i\varepsilon}\right]_{\alpha\beta} e^{-ip(x-y)}$$

Elettrodinamica Quantistica - QED

$$\mathcal{L}^{QED} = \bar{\Psi}\left(i\not{D} - m\right)\psi - \frac{1}{4}F_{\mu\nu}^{2}$$

$$D_\mu = \partial_\mu - iQeA_\mu \qquad\qquad \psi \to e^{ieQ\alpha(x)}\psi$$

Invariante sotto: $A_\mu \to A_\mu + \partial_\mu\alpha(x)$, $\quad \bar{\psi} \to e^{-ieQ\alpha}\psi$

$$\partial_\mu\psi \to i e^{ieQ\alpha}\left(\partial_\mu\psi + ieQ\alpha\psi\right), \qquad D_\mu\psi \to e^{ieQ\alpha(x)}D_\mu\psi$$

Gauge-fixing: $\mathcal{L}_{g.f.} = -\frac{1}{2\xi}\left(\partial_\mu A_\mu\right)^2$

Interazioni

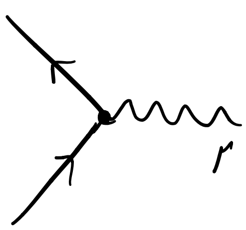
$$\mathcal{L}_{int}(A_\mu,\psi,\bar{\psi}) = eQ\,\bar{\psi}\gamma^\mu\psi A_\mu$$

Per l'elettrone $Q = -1$.

$$Z[J_\mu,\bar{\eta},\eta] = \int DA_\mu D\bar{\psi} D\psi \exp\left[i\int d^4x\left(\mathcal{L}^{QED} + \mathcal{L}_{g.f.} + J_\mu A^\mu + \bar{\eta}\psi + \bar{\psi}\eta\right)\right]$$

$$= Z_0[0]\, e^{i\left\langle \mathcal{L}_{int}\left(-i\frac{\delta}{\delta J_\mu}, -i\frac{\delta}{\delta \bar{\eta}}, i\frac{\delta}{\delta \eta}\right)\right\rangle}\; e^{-\frac{1}{2}\left\langle J_\mu(x)\, i\Pi^{\mu\nu}(x-y)\, J_\nu(y)\right\rangle}\; e^{-\left\langle \bar{\eta}(x)\, S_F(x-y)\, \eta(y)\right\rangle}$$

Regola di Feynman



$$= i\frac{\delta}{\delta\psi}\frac{\delta}{\delta\bar{\psi}}\frac{\delta}{\delta A_\mu}\left[eQ\,\bar{\psi}\gamma^\mu\psi A_\mu\right] = ieQ\,\gamma^\mu$$

QED SCALARE

- Campo scalare complesso + fotone

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \varphi)^* (D^\mu \varphi) - m^2 \varphi^* \varphi - \lambda (\varphi^* \varphi)^2$$

Dove $\quad D_\mu \varphi = (\partial_\mu - i e Q A_\mu)\varphi$

ESERCIZIO:

Derivare $\mathcal{L}_{int}$ tra φ ed il fotone e le regole di Feynman associate.