Introduction to Quantum Information

Outline

1. Quantum Entanglement and Complete Positivity

2. Classical Coding Theorems

3. Quantum channels and Coding

4. Classical and Quantum Perceptrons

1.1. $\boxed{\text{Quantum States}}$

- S: finite or infinite dimensional quantum system
- $\mathcal{H}$: corresponding Hilbert space

• Example 1.1.1

$$\boxed{\begin{array}{l} \text{Finite level system}: \quad \mathcal{H} = \mathbb{C}^n \quad n = 2, 3, \dots \\ \text{Infinite dimensional system}: \quad \mathcal{H} = L^2(\mathbb{R}^n) \end{array}}$$

- <u>Vector states</u>: $|\psi\rangle \in \mathcal{H}$, $\langle\psi|\psi\rangle = \|\psi\|^2 = 1$
- <u>Observables</u>: $A = A^\dagger : \mathcal{H} \circlearrowleft$ hermitian linear operators
- <u>Mean values</u>: $\langle A \rangle_\psi = \langle \psi | A \psi \rangle$

- Definition 1.1.1. (Positive Operators)

  A is positive-semidefinite $(A \geq 0)$ $\Leftrightarrow$ $\langle \psi | A \psi \rangle \geq 0 \;\; \forall |\psi\rangle \in \mathcal{H}$.

- Exercise 1.1.1 : Prove that $A \geq 0 \Rightarrow A = A^{+}$

Use $\langle \psi | A \phi \rangle = \frac{1}{4}\Big(\langle \psi+\phi | A(\psi+\phi)\rangle - \langle \psi-\phi | A(\psi-\phi)\rangle + i\langle \psi - i\phi | A(\psi - i\phi)\rangle - i \langle \psi + i\phi | A(\psi + i\phi)\rangle \Big)$

and $\langle \chi | A^{+} \chi \rangle = \langle A\chi | \chi \rangle = \overline{\langle \chi | A \chi \rangle} = \langle \chi | A \chi \rangle$ (since $\langle \chi | A\chi \rangle \geq 0$)

- Exercise 1.1.2 : Prove that $A \geq 0 \Leftrightarrow \text{Spectrum}(A) \subseteq \mathbb{R}^{+}$

Assume discrete spectral decomposition : $A = \sum_{\alpha} a_{\alpha} |\alpha\rangle\langle\alpha|$, $\langle \alpha | \beta \rangle = \delta_{\alpha\beta}$

Choose $|\psi\rangle = |\alpha^{*}\rangle$: $\langle \psi | A \psi \rangle = a_{\alpha^{*}} \geq 0$

Vice versa : $a_{\alpha} \geq 0 \;\; \forall \alpha \Rightarrow \langle \psi | A \psi \rangle = \sum_{\alpha} a_{\alpha} |\langle \alpha | \psi \rangle|^{2} \geq 0 \;\; \forall |\psi\rangle \in \mathcal{H}$

- **Definition 1.1.2** (Trace)

$$\mathrm{Tr}(A) = \sum_i \langle \psi_i | A \psi_i \rangle \quad \forall \, \{\psi_i\} \text{ ONB in } \mathcal{H}.$$

— Properties

- <u>Basis independence</u>: $\{|\psi_i\rangle\}, \{|\phi_j\rangle\}$ ONB in $\mathcal{H}$, $\boxed{\sum_i |\psi_i\rangle\langle\psi_i| = \sum_j |\phi_j\rangle\langle\phi_j| = \mathbb{1}}$

Completeness

$$\sum_i \langle \psi_i | A \psi_i \rangle = \sum_{i,j,k} \langle \psi_i | \phi_j \rangle \langle \phi_j | A \phi_k \rangle \langle \phi_k | \psi_i \rangle$$

$$= \sum_{j,k} \sum_i \langle \phi_k | \psi_i \rangle \langle \psi_i | \phi_j \rangle \langle \phi_j | A \phi_k \rangle$$

$$= \sum_{j,k} \delta_{jk} \langle \phi_j | A \phi_k \rangle = \sum_j \langle \phi_j | A \phi_j \rangle$$

- $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ (Cyclicity)

$$\mathrm{Tr}(AB) = \sum_j \langle \psi_j | AB \psi_j \rangle = \sum_{j,k} \langle \psi_j | A \psi_k \rangle \langle \psi_k | B \psi_j \rangle$$

$$= \sum_{k,j} \langle \psi_k | B \psi_j \rangle \langle \psi_j | A \psi_k \rangle = \sum_k \langle \psi_k | BA \psi_k \rangle = \mathrm{Tr}(BA) \quad \text{(by completeness)}$$

- $\boxed{\text{Exercise 1.1.3}}$: Prove that $\mathrm{Tr}\big(|\phi\rangle\langle\psi|\big) = \langle\psi|\phi\rangle$

Use ONB $\{|\psi_j\rangle\}$ such that $|\psi_1\rangle = |\psi\rangle$, $\langle\psi_j|\psi\rangle = 0 \ \forall\ j \geq 2$

- $\boxed{\text{Exercise 1.1.4}}$: Prove that $\langle A\rangle_\psi = \langle\psi|A\psi\rangle = \mathrm{Tr}\big(P_\psi A\big) = \mathrm{Tr}(AP_\psi)$

where $P_\psi = |\psi\rangle\langle\psi|$

Use Exercise 1.1.3

— Pure state statistics : given a vector state $|\psi\rangle \in \mathcal{H}$ such that $\||\psi\rangle\|^2 = \langle\psi|\psi\rangle = 1$ the statistics of any observable $A = A^\dagger : \mathcal{H} \circlearrowleft$ is determined by the one-dimensional projection $P_\psi = |\psi\rangle\langle\psi| = P_\psi^\dagger = P_\psi^2$ through the moments

$$\langle\psi|\,A^n\psi\rangle = \mathrm{Tr}\big(P_\psi A^n\big)$$

- Example 1.1.2 : Variance

$$\boxed{\begin{aligned} \Delta^2_\psi A &:= \langle (A - \langle A\rangle_\psi)^2\rangle_\psi \\ &= \langle A^2\rangle_\psi - \langle A\rangle_\psi^2 \\ &= \mathrm{Tr}(P_\psi A^2) - \big(\mathrm{Tr}(P_\psi A)\big)^2 \end{aligned}}$$

- Definition 1.1.3 (Pure states)

Projections onto normalized vector states are called PURE STATES

- Definition 1.1.4. (Statistical Ensembles)

Statistical ensembles are sets of vectors $\boxed{|\psi_j\rangle}$ with associated weights $\boxed{\alpha_j \geq 0}$ such that $\boxed{\sum_j \alpha_j = 1}$

$\{|\psi_j\rangle, \alpha_j\}$

Remark : Statistical ensembles correspond to the physical situation whereby one does only know that the system pure state appears in the set $\{|\psi_j\rangle\langle\psi_j|\}$ with a statistical weight α_j.

- Mean values with respect to statistical ensembles

$$\{|\psi_j\rangle, \alpha_j\} =: \mathcal{E} \ : \quad \langle A\rangle_{\mathcal{E}} = \sum_j \alpha_j \langle \psi_j | A \psi_j\rangle$$

$$= \sum_j \alpha_j \operatorname{Tr}\left(|\psi_j\rangle\langle\psi_j| A\right)$$

(Trace is linear: prove it) $$= \operatorname{Tr}\left(\left(\sum_j \alpha_j |\psi_j\rangle\langle\psi_j|\right) A\right) =: \langle A\rangle_\rho$$

- Definition 1.1.5 (Density matrices)

$\rho := \sum_j \alpha_j |\psi_j\rangle\langle\psi_j|$ is called <u>density matrix</u> or <u>mixed state</u>

$\langle A\rangle_\rho = \operatorname{Tr}(\rho A)$

— Properties

- $\operatorname{Tr} \rho = \sum_j \alpha_j \langle\psi_j|\psi_j\rangle = \sum_j \alpha_j = 1$ (normalization)
- $\langle\psi|\rho\psi\rangle = \sum_j \alpha_j |\langle\psi|\psi_j\rangle|^2 \geq 0$ (positivity)
- $\rho = \rho^\dagger$

Remark

- Given $\rho = \sum_j d_j |\psi_j\rangle\langle\psi_j|$, the weights $d_j \geq 0$ are probabilities of finding S described by ρ in the states $|\psi_j\rangle$ if and only if the $|\psi_j\rangle$ are orthonormal and the d_j are the eigenvalues of ρ.

  In general, $\langle\psi_k|\rho\,\psi_k\rangle = \sum_j d_j |\langle\psi_k|\psi_j\rangle|^2 \neq d_k$

- $\rho = \rho^\dagger = \sum_j d_j |\psi_j\rangle\langle\psi_j|$ can always be diagonalized

  $\rho = \sum_k r_k |r_k\rangle\langle r_k|$: $r_k \geq 0$, $\sum_k r_k = 1$, $\langle r_k | r_\ell\rangle = \delta_{k\ell}$

  Both statistical ensembles $\mathcal{E}_\rho = \{|\psi_j\rangle, d_j\}$ and $\mathcal{E}^{d}_\rho = \{|r_k\rangle, r_k\}$ are described by the same density matrix.

- **Definition 1.1.6**

  The space of states of a quantum system S is the set of all convex combination of its pure states.

- **Exercise 1.1.5** : prove that the convex combination of density matrices is still a density matrix

It suffices to consider $\rho = \mu \rho_1 + (1-\mu)\rho_2$ for $\rho_{1,2}$ two density matrices and $0 \le \mu \le 1$, then $\rho = \sum_j \mu\, d_{1j} |\psi_{1j}\rangle\langle\psi_{1j}| + \sum_k (1-\mu)\, d_{2k} |\psi_{2k}\rangle\langle\psi_{2k}|$

with weights $\mu d_{1j} \geq 0$ $(1-\mu) d_{2k} \geq 0$ such that $\sum_j \mu d_{1j} + \sum_k (1-\mu) d_{2k} =$

$$= \mu \sum_j d_{1j} + (1-\mu) \sum_k d_{2k} =$$

$$= \mu + (1-\mu) = 1$$

- **Exercise 1.1.6** : prove that ρ is a pure state iff $\rho^2 = \rho$

$\rho = P_\psi = |\psi\rangle\langle\psi| = P_\psi^2 = \rho^2$; let ρ be spectralized : $\rho = \sum_k r_k |r_k\rangle\langle r_k|$

$\rho = \rho^2 = \sum_k r_k |r_k\rangle\langle r_k| = \sum_k r_k^2 |r_k\rangle\langle r_k| \iff r_k = r_k^2$. Then use $\mathrm{Tr}\,\rho = 1$.

- $\boxed{\text{Exercise 1.1.7}}$: ρ is a pure state iff $\mathrm{Tr}(\rho^2) = 1$

$\rho = |\psi\rangle\langle\psi| \Longrightarrow \rho^2 = |\psi\rangle\langle\psi| \Longrightarrow \mathrm{Tr}(\rho^2) = 1$

Using the spectral decomposition of ρ : $\rho = \sum_k r_k |r_k\rangle\langle r_k|$, $\rho|r_k\rangle = r_k|r_k\rangle$

and $\mathrm{Tr}\,\rho = 1$, $\mathrm{Tr}(\rho^2) = \sum_k r_k^2 = 1 = \sum_k r_k \Rightarrow \sum_k r_k(1-r_k) = 0$

$\Rightarrow r_k(1-r_k) = 0 \quad \forall k$

- $\boxed{\text{Example 1.1.3}}$: q-bit , $\mathcal{H} = \mathbb{C}^2$, $\rho = \begin{pmatrix} r & s \\ \bar{s} & 1-r \end{pmatrix} = \begin{pmatrix} r & s_1 + i s_2 \\ s_1 - i s_2 & 1-r \end{pmatrix}$ $r, s_1, s_2 \in \mathbb{R}$

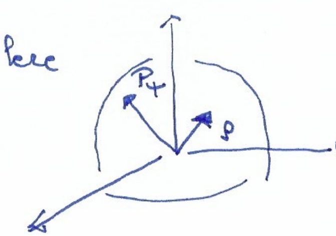
$\mathrm{Tr}\,\rho = 1$, positivity $\Longleftrightarrow r(1-r) \geq |s|^2 \Longleftrightarrow \mathrm{Det}\,\rho \geq 0 \Longleftrightarrow 0 \leq r \leq 1$, $r(1-r) \geq |s|^2$

<u>Bloch sphere</u> : Pauli matrices $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\rho = \frac{1 + \vec{r}\cdot\vec{\sigma}}{2} = \frac{1}{2}\left(1 + r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3\right) \qquad \begin{cases} r_1 = 2s_1 \\ r_2 = -2s_2 \\ r_3 = 2r-1 \end{cases}$$

$\mathrm{Det}\,\rho \geq 0 \Longleftrightarrow r_1^2 + r_2^2 + r_3^2 = \|\vec{r}\|^2 \leq 1$ inside of the Bloch-sphere

$P_\psi = \rho^2 = \rho \Longleftrightarrow \mathrm{Det}\,\rho = 0 \Longleftrightarrow \|\vec{r}\| = 1$ surface of the Bloch sphere



1.2. $\boxed{\text{Bipartite Entanglement}}$

$$S_1 + S_2$$

— Compound systems (Alice and Bob) : $\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$

$$\mathcal{H}_{12} \ni |\psi\rangle = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} c_{ij} \, |\psi_{1i}\rangle \otimes |\psi_{2j}\rangle \; ; \quad d_{1,2} = \dim(\mathcal{H}_{12})$$

$\{|\psi_{1i}\rangle\}_i$ ONB in $\mathcal{H}_1$; $\{|\psi_{2j}\rangle\}_j$ ONB in $\mathcal{H}_2$

$\boxed{\text{. Definition 1.2.1}}$: Partial Trace

$A : \mathcal{H}_{12} \circlearrowleft$

$$\mathrm{Tr}_1 : A \longrightarrow \sum_i \langle \psi_{1i} | A | \psi_{1i} \rangle : \mathcal{H}_2 \circlearrowleft$$

$$\mathrm{Tr}_2 : A \longrightarrow \sum_j \langle \psi_{2j} | A | \psi_{2j} \rangle : \mathcal{H}_1 \circlearrowleft$$

• $\boxed{\text{Exercise 1.2.1}}$: prove independence from ONB

Use Property 1 in page 3

– Single system observables $A_1 \otimes \mathbb{1}$, $\mathbb{1} \otimes A_2$

$\boxed{\text{· Definition 1.2.2}}$ <u>Local observables</u>

$A_1 \otimes A_2$

$\boxed{\text{· Definition 1.2.3}}$ <u>Reduced (marginal) states</u>

$\rho_{12} \in \mathcal{S}(S_1 + S_2)$: $\boxed{\rho_1 := \mathrm{Tr}_2\, \rho_{12}}$; $\boxed{\rho_2 := \mathrm{Tr}_1\, \rho_{12}}$

<u>Remark</u> : Physical meaning of reduced states

$$\mathrm{Tr}\left(\rho_{12} A_1 \otimes \mathbb{1}\right) = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} \langle \psi_{1i} \otimes \psi_{2j} | \rho_{12} | A_1 \psi_{1i} \otimes \psi_{2j} \rangle$$

$$= \sum_{i=1}^{d_1} \langle \psi_{1i} | \left(\sum_{j=1}^{d_2} \langle \psi_{2j} | \rho_{12}\, \psi_{2j} \rangle \right) A_1 | \psi_{1i} \rangle$$

$$= \mathrm{Tr}_1 \left(\rho_1 A_1 \right)$$

$$\mathrm{Tr}\left(\rho_{12}\, \mathbb{1} \otimes A_2\right) = \mathrm{Tr}_2 \left(\rho_2 A_2 \right)$$

$\boxed{\cdot\ \text{Theorem } 1.2.1.}$ Schmidt Decomposition

$$\mathcal{H}_{12} \ni |\psi\rangle_{12} = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} c_{ij}\, |\psi_{1i}\rangle \otimes |\psi_{2j}\rangle = \boxed{\sum_{k=1}^{\min(d_1,d_2)} \sqrt{\mu_k}\, |\phi_{1k}\rangle \otimes |\phi_{2k}\rangle}$$

$$\mu_k \geq 0 \ ; \quad \langle \phi_{1k} | \phi_{1\ell} \rangle = \langle \phi_{2k} | \phi_{2\ell} \rangle = \delta_{k\ell}$$

Proof. $d_1 \leq d_2$

$$|\psi\rangle_{12} = \sum_{i=1}^{d_1} |\psi_{1i}\rangle \otimes \Big(\underbrace{\sum_{j=1}^{d_2} c_{ij} |\psi_{2j}\rangle}_{|\tilde{\psi}_{2i}\rangle} \Big) = \sum_{i=1}^{d_1} |\psi_{1i}\rangle \otimes |\tilde{\psi}_{2i}\rangle$$

$$|\psi\rangle_{12}\langle\psi|_{12} = \sum_{i,j=1}^{d_1} |\psi_{1i}\rangle\langle\psi_{1j}| \otimes |\tilde{\psi}_{2i}\rangle\langle\tilde{\psi}_{2j}|$$

Spectral decomposition

$$\rho_1 := \mathrm{Tr}_2\big(|\psi\rangle_{12}\langle\psi|_{12}\big) = \sum_{i,j=1}^{d_1} |\psi_{1i}\rangle\langle\psi_{1j}|\, \langle\tilde{\psi}_{2j}|\tilde{\psi}_{2i}\rangle = \boxed{\sum_{k=1}^{d_1} \mu_k |\phi_{1k}\rangle\langle\phi_{1k}|}$$

– Choose $|\psi_{1i}\rangle = |\phi_{1i}\rangle$, then $\langle\tilde{\psi}_{2j}|\tilde{\psi}_{2i}\rangle = \mu_i \delta_{ij}$

– Set $|\phi_{2i}\rangle := \dfrac{|\tilde{\psi}_{2i}\rangle}{\sqrt{\mu_i}}$, $\mu_i \neq 0$

– Then $|\psi\rangle_{12} = \sum_{i=1}^{d_1} \sqrt{\mu_i}\, |\phi_{1i}\rangle \otimes |\phi_{2i}\rangle$

Remark

- The Schmidt coefficients μ_k are the eigenvalues of the reduced density matrix $\rho_1 = \mathrm{Tr}_2\, \rho_{12}$, $\rho_{12} = |\psi\rangle_{12}\langle\psi|_{12}$ and the vectors $|\phi_{1k}\rangle$ its eigenvectors

- $\rho_2 := \mathrm{Tr}_1\, \rho_{12}$ has eigenvalues μ_k and eigenvectors $|\phi_{2k}\rangle$

Proof:
$$|\psi\rangle_{12}\langle\psi|_{12} = \sum_{i,j=1}^{d_1} \sqrt{\mu_i \mu_j}\, |\phi_{1i}\rangle\langle\phi_{1j}| \otimes |\phi_{2i}\rangle\langle\phi_{2j}|$$

$$\rho_2 = \sum_{i,j=1}^{d_1} \sqrt{\mu_i \mu_j}\, \langle\phi_{1j}|\phi_{1i}\rangle\, |\phi_{2i}\rangle\langle\phi_{2j}| = \sum_{i=1}^{d_1} \mu_i\, |\phi_{2i}\rangle\langle\phi_{2i}|$$

$\boxed{\text{• Proposition 1.2.1}}$: $|\psi_{12}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ iff $\rho_1^2 = \rho_1$ $(\rho_2^2 = \rho_2)$

$\boxed{\text{Proof}}$
$$|\psi_{12}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \implies \rho_1 = |\psi_1\rangle\langle\psi_1| = \rho_1^2$$

$$\rho_1^2 = \rho_1 \implies \rho_1 = |\psi_1\rangle\langle\psi_1| \implies \mu_1 = 1,\ \mu_i = 0\ \forall i \geq 2 \implies |\psi_{12}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$

. Definition 1.2.4

Bipartite state vectors $|\psi_{12}\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ are called $\boxed{\text{separable}}$ if they can be written in the form $\quad |\psi_{12}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$

otherwise they are called $\boxed{\text{entangled}}$.

Bipartite mixed states $\rho_{12} : \mathcal{H}_{12} \circlearrowleft$ are called $\boxed{\text{separable}}$ if they can be written as linear convex combinations of tensor products of density matrices $\rho_{1i} \in \mathcal{S}(S_1)$ and $\rho_{2j} \in \mathcal{S}(S_2)$

$$\rho_{12} = \sum_{i,j} d_{ij} \, \rho_{1i} \otimes \rho_{2j} \quad ; \quad d_{ij} \geq 0 \, , \ \sum_{i,j} d_{ij} = 1$$

Otherwise they are called $\boxed{\text{entangled}}$

$\boxed{\text{. Example 1.2.1.}}$ Two qubit Bell's basis

$\mathbb{C}^2 \ni \{|0\rangle, |1\rangle\}$ $\boxed{\begin{matrix}\sigma_z|0\rangle = |0\rangle \\ \sigma_z|1\rangle = -|1\rangle\end{matrix}}$ $\boxed{\begin{matrix}\sigma_x|0\rangle = |1\rangle \\ \sigma_x|1\rangle = |0\rangle\end{matrix}}$ $\boxed{\begin{matrix}\sigma_y|0\rangle = i|1\rangle \\ \sigma_y|1\rangle = -i|0\rangle\end{matrix}}$

$\boxed{\begin{matrix}\sigma_x, \sigma_y, \sigma_z \\ \text{Pauli matrices}\end{matrix}}$

uniform state vector $\boxed{|\Psi_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}}$

$|ij\rangle := |i\rangle \otimes |j\rangle$

$$|\Psi_{00}\rangle\langle\Psi_{00}| = \frac{1}{2}\Big[|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| + |0\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 0| \otimes |1\rangle\langle 0|\Big]$$

$$|0\rangle\langle 0| = \frac{1+\sigma_z}{2} \qquad |1\rangle\langle 1| = \frac{1-\sigma_z}{2} \qquad |0\rangle\langle 1| = \frac{\sigma_x + i\sigma_y}{2} \qquad |1\rangle\langle 0| = \frac{\sigma_x - i\sigma_y}{2}$$

$$|\Psi_{00}\rangle\langle\Psi_{00}| = \frac{1}{4}\Big[1 \otimes 1 + \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z\Big] = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\boxed{\rho_1 = \frac{1}{2}\Big[|0\rangle\langle 0| + |1\rangle\langle 1|\Big] = \rho_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

$$\begin{cases} |\Psi_{10}\rangle = \sigma_1 \otimes 1\, |\Psi_{00}\rangle = \dfrac{|01\rangle + |10\rangle}{\sqrt{2}} \\[2ex] |\Psi_{01}\rangle = \sigma_3 \otimes 1\, |\Psi_{00}\rangle = \dfrac{|00\rangle - |11\rangle}{\sqrt{2}} \\[2ex] |\Psi_{11}\rangle = i\sigma_2 \otimes 1\, |\Psi_{00}\rangle = \dfrac{|01\rangle - |10\rangle}{\sqrt{2}} \end{cases}$$

Remark : $\boxed{\text{NON-LOCALITY}}$ of $|\psi_{00}\rangle$ (1)

$\left\{|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}, |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}\right\}$ ONB in $\mathbb{C}^2$

Hadamard Gate : $H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = H^\dagger = H^{-1}$ $\quad \begin{cases} H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \\ H|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \end{cases}$

CNOT Gate : $U_{CNOT} = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \sigma_x = \begin{pmatrix}\mathbb{1} & 0\\ 0 & \sigma_x\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$

$U_{CNOT}^{\dagger} = U_{CNOT}^{-1}$

$$\begin{cases} U_{CNOT}|00\rangle = |00\rangle \\ U_{CNOT}|01\rangle = |01\rangle \\ U_{CNOT}|10\rangle = |11\rangle \\ U_{CNOT}|11\rangle = |10\rangle \end{cases}$$

target qubit

$|i\, j\rangle \xrightarrow[U_{CNOT}]{} |i\ j\oplus i\rangle$

control qubit

$H \otimes \mathbb{1}|00\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle$ $\qquad$ Separable state $\xrightarrow{\text{Local Operation}}$ Separable state

$U_{CNOT}\, H \otimes \mathbb{1}|00\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ $\qquad$ Separable state $\xrightarrow[\text{NON-LOCAL operation}]{}$ Entangled state

Remark : $\boxed{\text{NON-LOCALITY}}$ of $|\Psi_{00}\rangle$ (2)

Mean values of $\boxed{\text{local observables}}$ with respect to $\boxed{\text{separable states}}$ factorize:

$$\langle \psi_1 \otimes \psi_2 |\, A_1 \otimes A_2 \,| \psi_1 \otimes \psi_2 \rangle = \langle \psi_1 | A_1 \psi_1 \rangle \langle \psi_2 | A_2 \psi_2 \rangle$$

Consider $|\Psi_{00}\rangle = \dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$

$$\sigma_z \otimes \sigma_z \,|\Psi_{00}\rangle = |\Psi_{00}\rangle \quad ; \quad \sigma_z \otimes 1 \,|\Psi_{00}\rangle = \underbrace{\frac{|00\rangle - |11\rangle}{\sqrt{2}}}_{|\Psi_{01}\rangle \perp |\Psi_{00}\rangle} = 1 \otimes \sigma_z \,|\Psi_{00}\rangle$$

(see Example 1.2.1)

$$\langle \Psi_{00} | \sigma_z \otimes \sigma_z | \Psi_{00} \rangle = 1 \quad ; \quad \langle \Psi_{00} | \sigma_z \otimes 1 | \Psi_{00} \rangle = \langle \Psi_{00} | \Psi_{01} \rangle = \langle \Psi_{00} | 1 \otimes \sigma_z | \Psi_{00} \rangle = 0$$

$$\boxed{\langle \Psi_{00} | \sigma_z \otimes \sigma_z | \Psi_{00} \rangle \neq \langle \Psi_{00} | \sigma_z \otimes 1 | \Psi_{00} \rangle \langle \Psi_{00} | 1 \otimes \sigma_z | \Psi_{00} \rangle}$$

NON-LOCAL CORRELATIONS IN $|\Psi_{00}\rangle$

$\boxed{\cdot\ \text{Exercise 1.2.2}}$ Extrapolate the two qubit state $|\Psi_{00}\rangle$ to higher dimension.

$$\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2 \quad ; \quad \mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}^d \ni \{|i\rangle\}_{i=1}^{d} \ \text{ONB} \ , \ d \geq 3$$

$$|\Psi_{mif}\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle \quad , \quad |ii\rangle = |i\rangle \otimes |i\rangle$$

$$P_{mif} = |\Psi_{mif}\rangle\langle\Psi_{mif}| = \frac{1}{d} \sum_{i,j=1}^{d} |i\rangle\langle j| \otimes |i\rangle\langle j|$$

$$\rho_1 = \mathrm{Tr}_2\left(P_{mif}\right) = \frac{1}{d} \sum_{i=1}^{d} |i\rangle\langle i| = \frac{\mathbb{1}_d}{d} \quad \left(\text{by completeness of } \{|i\rangle\}_{i=1}^{d}\right)$$

$$\rho_2 = \mathrm{Tr}_1\left(P_{mif}\right) = \frac{1}{d} \sum_{j=1}^{d} |j\rangle\langle j| = \frac{\mathbb{1}_d}{d} \qquad \mathbb{1}_d = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

<u>Remark</u>

$\frac{\mathbb{1}_d}{d}$ is known as <u>maximally mixed</u> or <u>totally depolarized</u> state

$\boxed{\cdot\ \text{Exercise 1.2.3}}$: try to justify the latter denomination from a physical point of view.