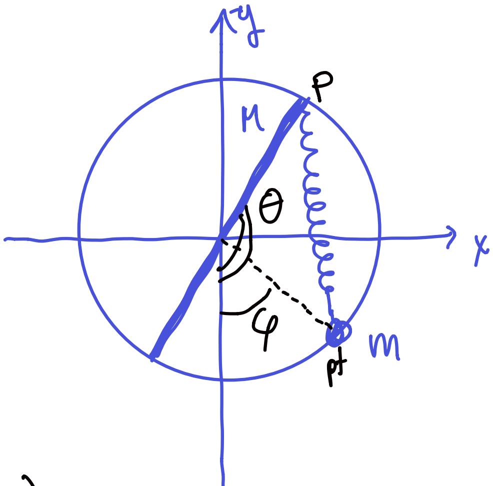
ESERCIZI

ES. 2 del 26.07.18

$$I = \frac{Md^2}{12} \quad \text{dove } d = 2R$$

$$= \frac{M\,4R^2}{12} = \frac{MR^2}{3}$$



$\downarrow g$

$x_{pt} = R \operatorname{sen}\varphi \qquad \dot{x}_{pt} = R\dot{\varphi}\cos\varphi$

$y_{pt} = -R\cos\varphi \qquad \dot{y}_{pt} = R\dot{\varphi}\operatorname{sen}\varphi$

$$I = \frac{MR^2}{3}$$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \operatorname{sen}\alpha\operatorname{sen}\beta$

1) $L(\varphi, \theta, \dot{\varphi}, \dot{\theta}) = T - V$

$$T = \frac{1}{2} m R^2 \dot{\varphi}^2 + \frac{1}{2} I \dot{\theta}^2 \qquad V = mg\,y_{pt} + \frac{1}{2} K d^2$$

$x_p = R\operatorname{sen}\theta \qquad y_p = -R\cos\theta$

$$d^2 = (x_{pt} - x_p)^2 + (y_{pt} - y_p)^2 = (R\operatorname{sen}\varphi - R\operatorname{sen}\theta)^2 + (-R\cos\varphi + R\cos\theta)^2$$

$$= R^2(\operatorname{sen}^2\varphi + \operatorname{sen}^2\theta - 2\operatorname{sen}\varphi\operatorname{sen}\theta + \cos^2\varphi + \cos^2\theta - 2\cos\varphi\cos\theta)$$

$$= R^2(2 - 2\cos(\varphi - \theta))$$

$$L = \frac{1}{2} mR^2\dot{\varphi}^2 + \frac{1}{2} I\dot{\theta}^2 + mgR\cos\varphi - KR^2(1 - \cos(\varphi - \theta))$$

2) Ep. del sist. e stab.

$$V(\theta, \varphi) = -mgR\cos\varphi + KR^2(1 - \underbrace{\cos(\varphi - \theta)}_{=\cos(\theta - \varphi)})$$

$$0 = \frac{\partial V}{\partial \theta} = KR^2\operatorname{sen}(\theta - \varphi) \rightarrow \operatorname{sen}(\theta - \varphi) = 0 \rightarrow \theta - \varphi = n\pi \quad n = 0, 1$$

$$0 = \frac{\partial V}{\partial \varphi} = mgR\operatorname{sen}\varphi + KR^2\cancel{\operatorname{sen}(\varphi - \theta)} \rightarrow \operatorname{sen}\varphi = 0 \rightarrow \varphi = \ell\pi \quad \ell = 0, 1$$

$$(\theta^*, \varphi^*) = (0, 0), (\pi, 0), (\pi, \pi), (0, \pi)$$

$$\mathcal{B}(\theta,\varphi) = \frac{\partial^2 V}{\partial q_i \partial q_j} = \begin{pmatrix} KR^2\cos(\theta-\varphi) & -KR^2\cos(\theta-\varphi) \\ -KR^2\cos(\theta-\varphi) & mgR\cos\varphi + KR^2\cos(\theta-\varphi) \end{pmatrix}$$

$q_1 = \theta$

$q_2 = \varphi$

Pto ep. è stabile se e solo se l'Hessiano $\mathcal{B}$ è una matrice DEFINITA POSITIVA!

$$\mathcal{B}(0,0) = \begin{pmatrix} KR^2 & -KR^2 \\ -KR^2 & mgR + KR^2 \end{pmatrix}$$

$\det = KR^2(\cancel{KR^2} + mgR) - (\cancel{KR^2})^2 = mgKR^3 > 0$

$\mathrm{Tr} = 2KR^2 + mgR > 0$ $\Rightarrow$ def. pos.

[Matrice M 2×2 è def. positiva se i suoi autovalori sono entrambi positivi $\lambda_1 > 0$ $\lambda_2 > 0$ $\iff$ $\det M = \lambda_1 \cdot \lambda_2 > 0$ & $\mathrm{tr} M = \lambda_1 + \lambda_2 > 0$]

$\Rightarrow$ $(0,0)$ è pto ep. STABILE

$\mathrm{tr}(ABC) = \mathrm{tr}(CAB)$

$\mathrm{tr}(O^T M O) = \mathrm{tr}(O O^T M) = \mathrm{tr}(M)$

$$\mathcal{B}(\pi,0) = \begin{pmatrix} -KR^2 & KR^2 \\ KR^2 & mgR - KR^2 \end{pmatrix}$$

$\det = KR^2(KR^2 - mgR) - (KR^2)^2 \overset{!}{=} -KR^3 mg < 0$

$\Rightarrow$ $(\pi,0)$ è pto ep. INSTABILE

$$\mathcal{B}(0,\pi) = \begin{pmatrix} -KR^2 & KR^2 \\ KR^2 & -mgR - KR^2 \end{pmatrix}$$

$\mathrm{tr} = -2KR^2 - mgR < 0$

$\Rightarrow$ $(0,\pi)$ è pto ep. INSTABILE

$$\mathcal{B}(\pi,\pi) = \begin{pmatrix} KR^2 & -KR^2 \\ -KR^2 & -mgR + KR^2 \end{pmatrix}$$

$\det = KR^2(KR^2 - mgR) - (KR^2)^2 = -mgKR^3 < 0$

$\Rightarrow$ (π,π) è pto ep. INSTABILE

3) $\hat{L}$ attorno a pto ep. stab. $(0,0)$

$$B = \mathcal{B}(0,0) = \begin{pmatrix} KR^2 & -KR^2 \\ -KR^2 & KR^2 + mgR \end{pmatrix} \qquad A = \mathcal{A}(0,0) = \begin{pmatrix} I & \\ & mR^2 \end{pmatrix}$$

$$\hat{L} = \frac{1}{2}(\dot\theta,\dot\varphi)\begin{pmatrix} I & \\ & mR^2\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\varphi\end{pmatrix} - \frac{1}{2}(\theta,\varphi)\begin{pmatrix} kR^2 & -kR^2 \\ -kR^2 & kR^2+mgR\end{pmatrix}\begin{pmatrix}\theta\\ \varphi\end{pmatrix}$$

$$= \frac{1}{2}I\dot\theta^2 + \frac{1}{2}mR^2\dot\varphi^2 - \frac{1}{2}kR^2\theta^2 - \frac{1}{2}(kR^2+mgR)\varphi^2 - kR^2\theta\varphi$$

$$\begin{pmatrix} I & \\ & mR^2\end{pmatrix}\begin{pmatrix}\ddot\theta\\ \ddot\varphi\end{pmatrix} + \begin{pmatrix} kR^2 & -kR^2 \\ -kR^2 & kR^2+mgR\end{pmatrix}\begin{pmatrix}\theta\\ \varphi\end{pmatrix} = 0$$

4) $M = 3m \Rightarrow I = mR^2$ e determiniamo freq. piccole osc.

$$\det(B - \lambda A) = 0$$

$$\det\begin{pmatrix} kR^2-\lambda mR^2 & -kR^2 \\ -kR^2 & kR^2+mgR-\lambda mR^2\end{pmatrix} =$$

$$= \underbrace{\det \cancel{mR^2}\begin{pmatrix} \frac{k}{m}-\lambda & -\frac{k}{m} \\ -k/m & g/R + k/m - \lambda\end{pmatrix}}$$

$$= \left(\lambda - \frac{k}{m}\right)\left(\lambda - \frac{k}{m} - \frac{g}{R}\right) - \left(\frac{k}{m}\right)^2$$

$$\lambda^2 - \left(\frac{2k}{m} + \frac{g}{R}\right)\lambda + \frac{k}{m}\frac{g}{R} = 0$$

vediamo che $\lambda_{1,2} > 0$ come ci aspettiamo per pto eq. stab.

$$\lambda_{1,2} = \left(\frac{k}{m} + \frac{g}{2R}\right) \pm \sqrt{\underbrace{\left(\frac{k}{m}+\frac{g}{2R}\right)^2 - \frac{k}{m}\frac{g}{R}}_{\left(\frac{k}{m}\right)^2 + \left(\frac{g}{2R}\right)^2}}$$

$$\lambda_{1,2} = \omega_{1,2}^2$$

5) $g=0$, c'è cost. del moto?

$$L = \frac{1}{2} mR^2\dot\varphi^2 + \frac{1}{2} I\dot\theta^2 + \cancel{mgR\cos\varphi} - KR^2(1-\cos(\varphi-\theta))$$

$\Downarrow$

L è invariante per la trasf.

$\theta \mapsto \theta + \alpha$

$\varphi \mapsto \varphi + \alpha$

$\Downarrow$

$\exists$ cost. del moto $\rightarrow$ per Nöther

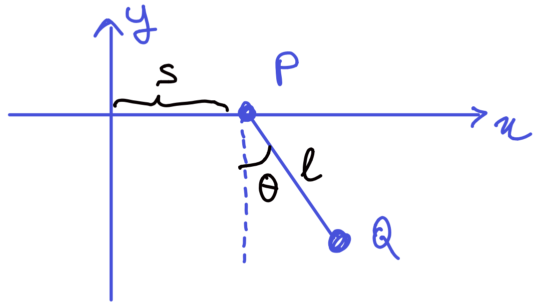
$$P = \sum_i \frac{\partial L}{\partial \dot q_i} \frac{\partial \varphi_i}{\partial \alpha} = \underbrace{I\dot\theta + mR^2\dot\varphi}$$

componente del mom. ang. lungo asse z

$\downarrow$

Infatti il sist. è INV. PER ROTAZIONI attorno asse z (se $g=0$)

Es. 2 del 24.06.19



$\downarrow \vec{g}$

$x_P = s \qquad \dot{x}_P = \dot{s}$

$y_P = 0 \qquad \dot{y}_P = 0$

$x_Q = s + l \,\text{sen}\,\theta \qquad \dot{x}_Q = \dot{s} + l\dot{\theta}\cos\theta$

$y_Q = -l\cos\theta \qquad \dot{y}_Q = l\dot{\theta}\,\text{sen}\,\theta$

1) $L = T - V$

$$T = \frac{m_P}{2}(\dot{x}_P^2 + \dot{y}_P^2) + \frac{m_Q}{2}(\dot{x}_Q^2 + \dot{y}_Q^2) =$$

$$= \frac{m_P}{2}\dot{s}^2 + \frac{m_Q}{2}\left(\dot{s}^2 + 2l\dot{s}\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right)$$

$M \equiv m_P + m_Q$

$m = m_Q$

$$= \frac{1}{2}\underbrace{(m_P + m_Q)}\dot{s}^2 + \frac{m_Q l^2\dot{\theta}^2}{2} + m_Q l\dot{s}\dot{\theta}\cos\theta$$

$$= \frac{1}{2}M\dot{s}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + ml\dot{s}\dot{\theta}\cos\theta$$

$$V = m_Q g y_Q = -mgl\cos\theta$$

$$L = \frac{1}{2}M\dot{s}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + ml\dot{s}\dot{\theta}\cos\theta + mgl\cos\theta$$

$\frac{1}{2}\cdot 2$

2) $T = \frac{1}{2}\sum_{h,k=1}^{2} \underline{a_{hk}(q)}\,\dot{q}_h\dot{q}_k$

$$a(\theta) = \begin{pmatrix} M & ml\cos\theta \\ ml\cos\theta & ml^2 \end{pmatrix}$$

3) $$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} = \frac{d}{dt}\left(M\dot{s} + ml\dot{\theta}\cos\theta\right) = M\ddot{s} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\,\text{sen}\,\theta$$

$$\frac{\partial L}{\partial s} = 0$$

$$M\ddot{s} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\,\text{sen}\,\theta = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}\left(ml^2\dot{\theta} + ml\dot{s}\cos\theta\right) = ml^2\ddot{\theta} + ml\ddot{s}\cos\theta - \cancel{ml\dot{s}\dot{\theta}}\,\text{sen}\,\theta$$

$$\frac{\partial L}{\partial \theta} = -ml\cancel{\dot{s}\dot{\theta}}\,\text{sen}\,\theta - mgl\,\text{sen}\,\theta$$

$$ml^2\ddot{\theta} + ml\ddot{s}\cos\theta + mgl\,\text{sen}\,\theta = 0$$

4) Coord. ciclica : s !

Cost. del moto è $\frac{\partial L}{\partial \dot s} = M\dot s + ml\dot\theta\cos\theta = p_s$

sist. inv. per traslaz. lungo x

$\Updownarrow$

cost. del moto è la componente della quantità di moto lungo x

5) $L^* = L - \dot s\, p_s \Big|_{\dot s = \dots}$

da inversione di $M\dot s + ml\dot\theta\cos\theta = \tilde p_s$

$\Updownarrow$

$$\dot s = \frac{\tilde p_s - ml\dot\theta\cos\theta}{M}$$

$$L = \frac{1}{2}M\dot s^2 + \frac{1}{2}ml^2\dot\theta^2 + ml\dot s\dot\theta\cos\theta + mgl\cos\theta$$

$$L^* = \frac{1}{2}M\left(\frac{\tilde p_s - ml\dot\theta\cos\theta}{M}\right)^2 + \frac{1}{2}ml^2\dot\theta^2 + ml\dot\theta\cos\theta\left(\frac{\tilde p_s - ml\dot\theta\cos\theta}{M}\right) + mgl\cos\theta - \tilde p_s\left(\frac{\tilde p_s - ml\dot\theta\cos\theta}{M}\right)$$

$$= \frac{1}{2}\frac{(\tilde p_s - ml\dot\theta\cos\theta)^2}{M} + \frac{1}{2}ml^2\dot\theta^2 - \frac{(\tilde p_s - ml\dot\theta\cos\theta)^2}{M} + mgl\cos\theta$$

$$= \frac{1}{2}ml^2\dot\theta^2 - \frac{(\tilde p_s - ml\dot\theta\cos\theta)^2}{2M} + mgl\cos\theta$$

$$= \frac{1}{2}ml^2\dot\theta^2 + mgl\cos\theta - \underbrace{\cancel{\frac{\tilde p_s^2}{2M}}}_{\text{cost.}} + \underbrace{\tilde p_s\frac{ml\dot\theta\cos\theta}{M}}_{\frac{d}{dt}\left[\frac{\tilde p_s ml}{M}\sin\theta\right]} - \frac{m^2l^2\dot\theta^2\cos^2\theta}{2M}$$

"derivata totale"

Espressione equivalente

$$L^* = \frac{1}{2}ml^2\dot\theta^2 + mgl\cos\theta - \frac{m^2l^2}{2M}\dot\theta^2\cos^2\theta$$

sist. ridotto a 1 grado di libertà (θ)

$$= \underbrace{\frac{1}{2}ml^2\dot\theta^2\left(1 - \frac{m}{M}\cos^2\theta\right)}_{T_{eff}} + \underbrace{mgl\cos\theta}_{-V_{eff}}$$

6) Pto equil. $V'_{eff} = 0 \leftrightarrow \operatorname{sen}\theta = 0 \qquad \theta = 0, \pi$

$V''_{eff}(0) = mgl\cos\theta \,\Big|_{\theta=0} = mgl > 0$

$(\theta = \pi \quad < 0)$

a) $\hat{L}^* = \frac{1}{2}\dot{\bar{q}}\cdot A\dot{\bar{q}} - \frac{1}{2}\bar{q}\cdot B\bar{q}$

$A = ml^2\left(1 - \frac{m}{M}\cos^2\theta\right)\Big|_{\theta=0} = ml^2\left(1 - \frac{m}{M}\right)$

$B = V''_{eff}(0) = mgl$

$\hat{L}^* = \frac{1}{2}ml^2\left(1 - \frac{m}{M}\right)\dot{\theta}^2 - \frac{1}{2}mgl\theta^2$

b) Esp. L^* attorno $\theta = 0$, tenendo solo termini quadratici:

$\frac{1}{2}ml^2\left(1 - \frac{m}{M}\right)\dot{\theta}^2 + mgl\left(\underbrace{\cancel{1}}_{\text{cost.}} - \frac{\theta^2}{2} + \dots\right)$

7) a) soluz. $\det(B - \lambda A) = 0 \leftrightarrow (mgl) - \lambda\left(ml^2\left(1 - \frac{m}{M}\right)\right)$

$\Rightarrow \omega^2 = \lambda = \dfrac{mgl}{ml^2\left(1 - \frac{m}{M}\right)} = \dfrac{g}{l}\dfrac{1}{1 - \frac{m}{M}} > 0$

b) Notiamo che $L_{o.a.} = \frac{1}{2}\mu\dot{x}^2 - \frac{1}{2}\mu\omega^2x^2$

$\omega^2 = \dfrac{\text{coeff } x^2}{\text{coeff } \dot{x}^2} = \dfrac{\frac{1}{2}mgl}{\frac{1}{2}ml^2\left(1 - \frac{m}{M}\right)} = \dfrac{g}{l}\dfrac{1}{1 - \frac{m}{M}}$