

SISTEMI DINAMICI

- LEZIONE DEL 23 APRILE 2020
 - PRIMA PARTE
-

Sistemi dinamici lineari

$$\dot{x} = f(t) \quad \rightarrow \quad \dot{\underline{x}} = \underline{Ax}$$

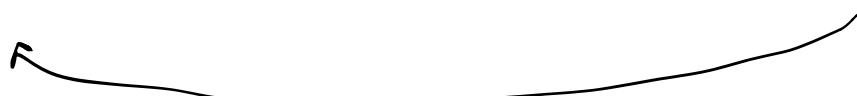
Problema spettrale per A

$$A\mathbf{v} = \lambda \mathbf{v} \quad \Rightarrow \quad \det(A - \lambda I) = 0$$

→ autovalori reali & distinti

→ autovalori complessi & distinti

$$\dot{x} = Ax \quad \sim \quad P^{-1}AP = B \quad \rightarrow \quad \dot{\underline{x}} = \underline{B}\underline{y}$$



A autovalori dist. ri, A T

$$T^{-1}AT = \begin{pmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & D_1 & \\ & & & \ddots & D_\ell \end{pmatrix}$$

$$D_i = \begin{pmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{pmatrix}$$

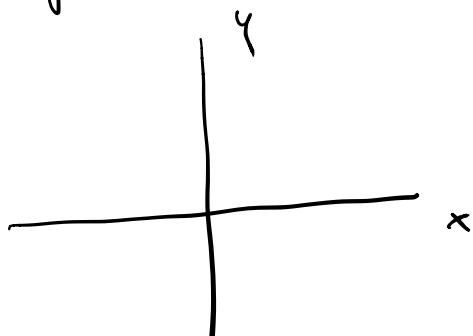
Analisi qualitativa per sistemi lineari

per la matrice

Consideriamo

$$\begin{cases} \dot{x} = a_{11}x + a_{12}y \\ \dot{y} = a_{21}y + a_{22}x \end{cases} \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

come sono fatte le proiettive nel piano delle fasi, piano- (x, y)



Lo carattere l'andamento delle proiettive è determinato da

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{a_{21}y + a_{22}x}{a_{11}x + a_{12}y}$$

Per capire meglio le traiettorie, consideriamo

le linee dove $\frac{dy}{dx}$ è costante, dette

ISOCLINEE (oppure NULLCLINE)

Ad esempio

$$\underline{\dot{x} = 0} \rightarrow \text{jettoni tangentii verticali (oppure stativi)}$$

$$\underline{\dot{y} = 0} \rightarrow \text{Tutti i vettori tangentii sono orizzontali}$$

(oppure stazionarie)

Audiamo a dominare : possibili componenti a seconda degli autovetori di A

Autovetori reali e distinti

Supponiamo $\lambda_1 < \lambda_2$. Abbiamo i casi
• $\lambda_1 < 0 < \lambda_2$, • $\lambda_1 < \lambda_2 < 0$, • $0 < \lambda_1 < \lambda_2$

Diagonaizzazione : $\begin{cases} \dot{x} = \lambda_1 x \\ \dot{y} = \lambda_2 y \end{cases}$

quindi $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{\lambda_2 y}{\lambda_1 x} \Rightarrow y^{\lambda_2} = K x^{\lambda_1}$

Esempi notevoli

$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \lambda_1 < 0 < \lambda_2 \quad \text{SELLA}$
(SADDLE)

Se soluzione è

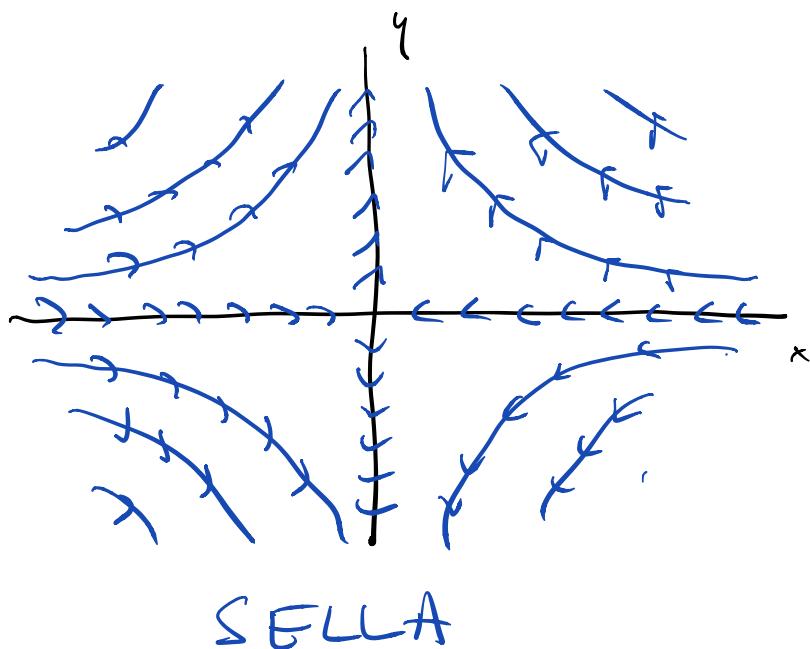
$$x(t) = \alpha e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\alpha e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ sulla am delle x e tende a $(0,0)$
per $t \rightarrow +\infty$ (perché $\lambda_1 < 0$)

Le chiamiamo "linee stabili"

$$\text{Si vede che } \lambda_2 > 0 \rightarrow \beta e^{\lambda_2 t} \left(\begin{matrix} 0 \\ 1 \end{matrix} \right)$$

\rightarrow le curve delle y e si allontanano da $(0,0)$ per $t \rightarrow +\infty$. Le chiamiamo "linee instabili".



Sol generica:

a $t \rightarrow +\infty$ sol va a infinito
in direzione delle linee instabili

($\alpha e^{\lambda_1 t} \rightarrow 0$
e $\beta e^{\lambda_2 t}$ domine)

Esempio $A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$ $\dot{x} = Ax$

$$\det \begin{pmatrix} 1-\lambda & 3 \\ 1 & -1-\lambda \end{pmatrix} = -(1-\lambda)(1+\lambda) - 3 = -1 + \lambda^2 - 3 = 0 \quad \lambda = \pm 2$$

$$\begin{pmatrix} 1-\lambda & 3 \\ 1 & -1-\lambda \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

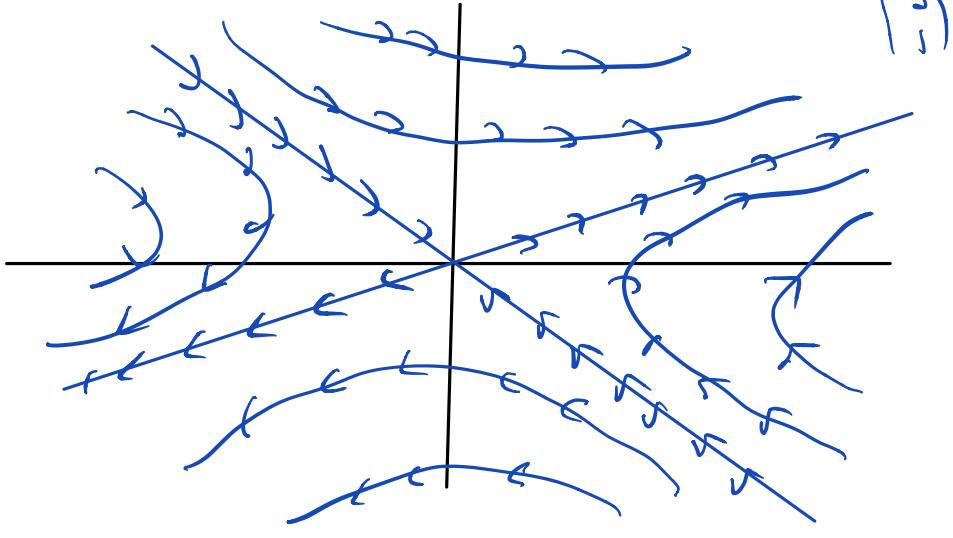
Si vede $\lambda = +2$ $\mathcal{S} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\lambda = -2$, $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$x(t) = \alpha e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \beta e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Rifatti
n fore

linee
insolubili
($T \rightarrow \infty$ via
dall'origine)

linee solubili
($T \rightarrow \infty$ verso
l'origine)



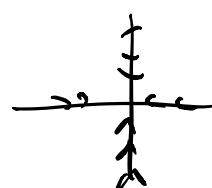
Esempio Pozzo (Sink)

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \lambda_1 < \lambda_2 < 0$$

Sol:

$$\mathbf{x}(T) = \alpha e^{\lambda_1 T} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda_2 T} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

entro in tendenza a $(0,0)$ per $T \rightarrow +\infty$
come si tendono?



Audiamo a vedere le

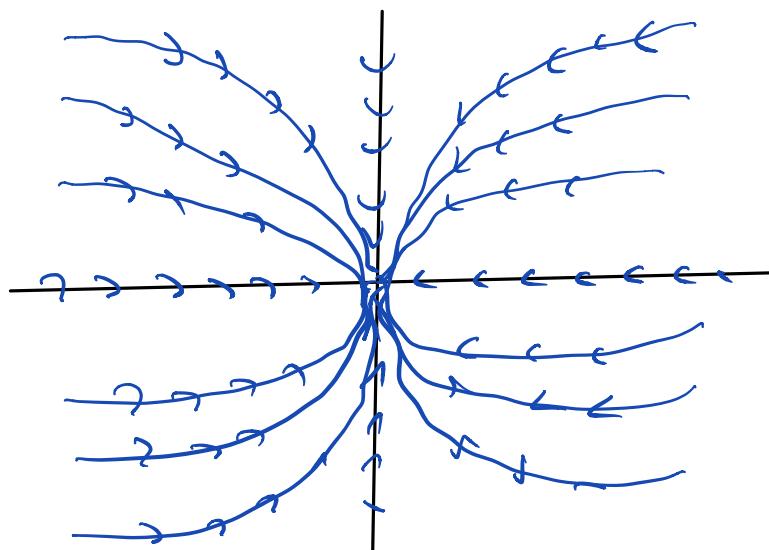
pendente $(\beta \neq 0)$

$$\frac{dy}{dx} = \frac{y}{x} = \frac{\lambda_2 \beta e^{\lambda_2 T}}{\lambda_1 \alpha e^{\lambda_1 T}} = \frac{\lambda_2 \beta}{\lambda_1 \alpha} e^{(\lambda_2 - \lambda_1)T}$$

$$\lambda_2 - \lambda_1 > 0$$

Per $t \rightarrow \infty$ le funzioni $\frac{dy}{dt}$ sono
 $\rightarrow \pm \infty \rightarrow$ verso l'origine in modo

Tangente all'asse y



$$\lambda_1 < \lambda_2$$

le componenti
 \rightarrow tende a
 zero più
 rapidamente
 della componenti
 y

Caso più generale

$$\alpha e^{\lambda_1 t} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \beta e^{\lambda_2 t} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\frac{dy}{dt} = \frac{\dot{y}}{x} = \frac{\lambda_1 \alpha e^{\lambda_1 t} u_2 + \lambda_2 \beta e^{\lambda_2 t} v_2}{\lambda_1 \alpha e^{\lambda_1 t} u_1 + \lambda_2 \beta e^{\lambda_2 t} v_1}$$

$$= \frac{\lambda_1 \alpha e^{(\lambda_1 - \lambda_2)t} u_2 + \lambda_2 \beta v_2}{\lambda_1 \alpha e^{(\lambda_1 - \lambda_2)t} u_1 + \lambda_2 \beta v_1}$$

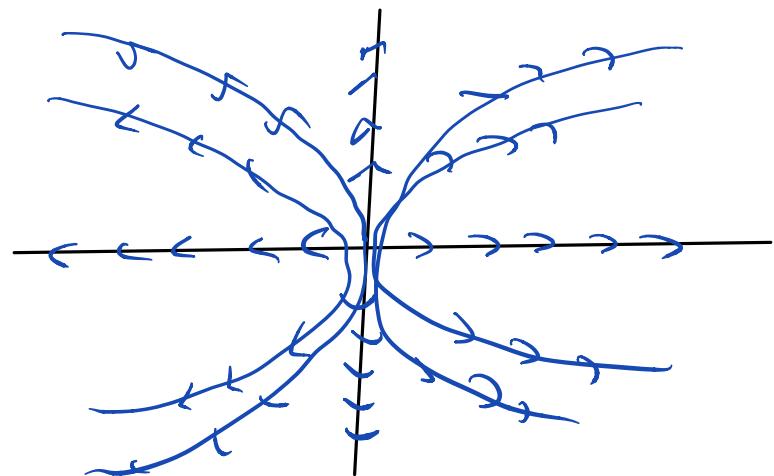
Siccome $\lambda_1 - \lambda_2 < 0$, per $t \rightarrow \infty$ tende alle
 proporzionali $\frac{v_2}{v_1}$ dell'autovettore λ_2
 $(\beta \neq 0)$

λ_2 è "più debole": le soluzioni tendono all'origine tangenzialmente alle linee soluzione, che compongono all'angolo
più debole

Esempio SORGENTI (SOURCÉ)

$$0 < \lambda_2 < \lambda_1$$

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$



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Autovetori complessi

Esempio $A = \begin{pmatrix} 0 & \beta \\ -\bar{\beta} & 0 \end{pmatrix}$ autovetori $\lambda = \pm i\beta$

Per $\lambda = i\beta$, si può verificare che $\begin{pmatrix} 1 \\ i \end{pmatrix}$ è

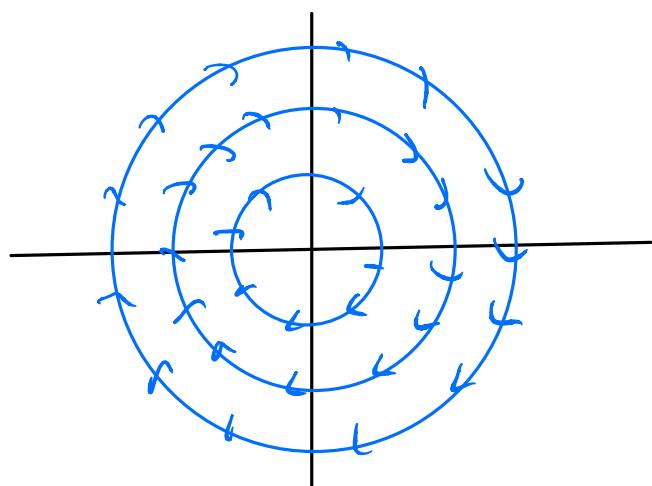
auto rettine

$$x(\tau) = c_1 \begin{pmatrix} \cos \beta \tau \\ -\sin \beta \tau \end{pmatrix} + c_2 \begin{pmatrix} \sin \beta \tau \\ \cos \beta \tau \end{pmatrix}$$

$$\Gamma_{x^2} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \rightarrow h_1 e^{\Gamma \tau} \cos \beta \tau + h_2 e^{\Gamma \tau} \sin \beta \tau$$

Le soluzioni sono periodiche o si perde

$\frac{2\pi}{\beta} \rightarrow$ tutte le soluzioni formano
cerchi attorno all'origine



OROGLIO $\beta > 0$

ANTIORO $\beta < 0$

CENTRO
(CENTER)

Esempio

SPIRALE (Ponza, Sogari)

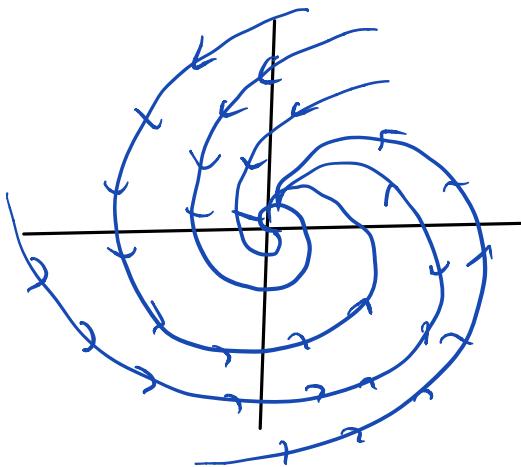
$$\begin{cases} \dot{x} = \alpha x + \beta y \\ \dot{y} = -\beta x + \alpha y \end{cases} \rightarrow$$

$$\begin{aligned} x &= r \cos \theta & : \begin{cases} \dot{r} = \alpha r \\ \dot{\theta} = -\beta \theta \end{cases} \\ y &= r \sin \theta \end{aligned}$$

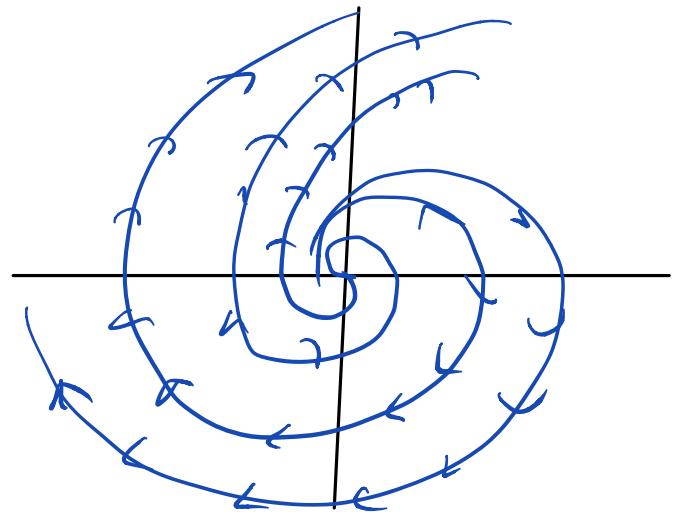
$$\lambda = \alpha \pm i\beta$$

Cose piane, ma $r = \alpha t$ converte i cerchi
in spirali, che entrano ($\alpha < 0$) o escono

$(\alpha > 0)$ dalla origine



$$\alpha < 0$$



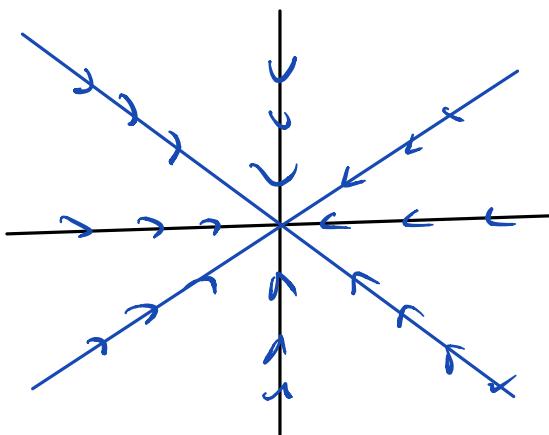
$$\alpha > 0$$

Autovalori ripetuti

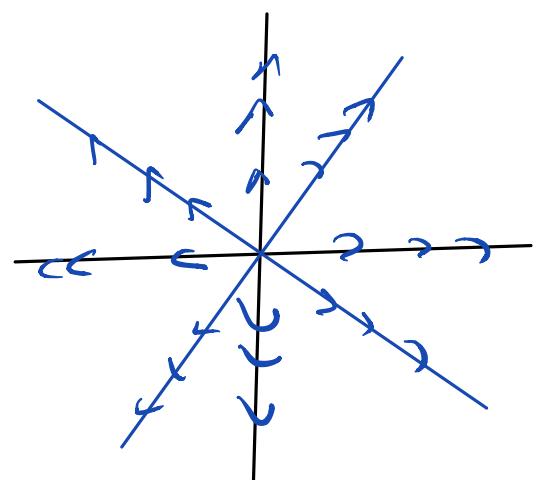
$$\lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1 = \lambda_2$$

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} : \begin{cases} x' = \lambda_1 x \\ y' = \lambda_1 y \end{cases} \quad y = kx$$

i punti $(0,0)$ e un modo



$$\lambda_1 < 0$$



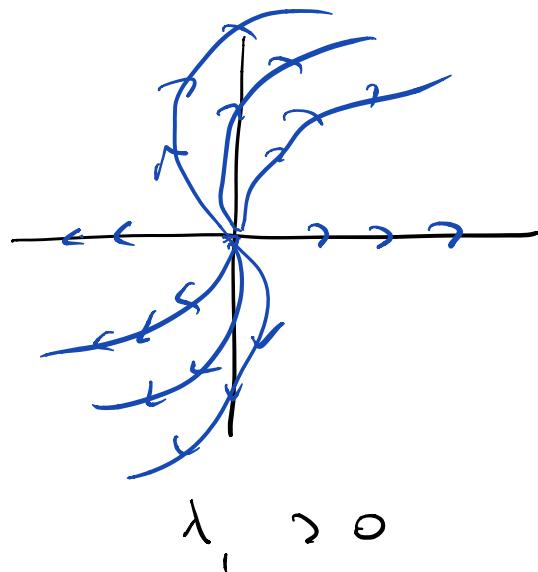
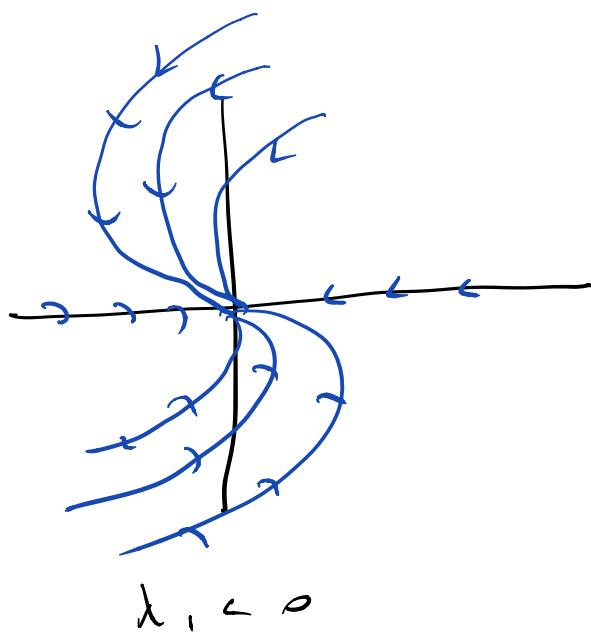
$$\lambda_1 > 0$$

$$A = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \quad \begin{cases} x' = \lambda_1 x + y \\ y' = \lambda_1 y \end{cases}$$

$$\Rightarrow \begin{cases} x = (\alpha + \beta \tau) e^{\lambda_1 \tau} \\ y = \beta e^{\lambda_1 \tau} \end{cases} \rightarrow \tau = \frac{1}{\lambda_1} \log \frac{y}{\beta}$$

Vediamo che $x = t y + \frac{\alpha}{\beta} y$

$$= \frac{y}{\lambda_1} (\log |y| + \text{cost.})$$



Esempio $t'' + 4t = 0$

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} x \rightsquigarrow t^2 + 4 = 0 \quad t = \pm 2i$$

$\lambda = +2i$ si verifica che $\begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{2it}$

è un autovettore $\begin{pmatrix} -2i & 1 \\ -4 & -2i \end{pmatrix} \begin{pmatrix} 1 \\ 2i \end{pmatrix} = 0$

$$e^{2it} \begin{pmatrix} 1 \\ 2i \end{pmatrix} \rightsquigarrow$$

$$X(\tau) = c_1 \begin{pmatrix} \cos 2\tau \\ -2 \sin 2\tau \end{pmatrix} + c_2 \begin{pmatrix} \sin 2\tau \\ 2 \cos 2\tau \end{pmatrix}$$

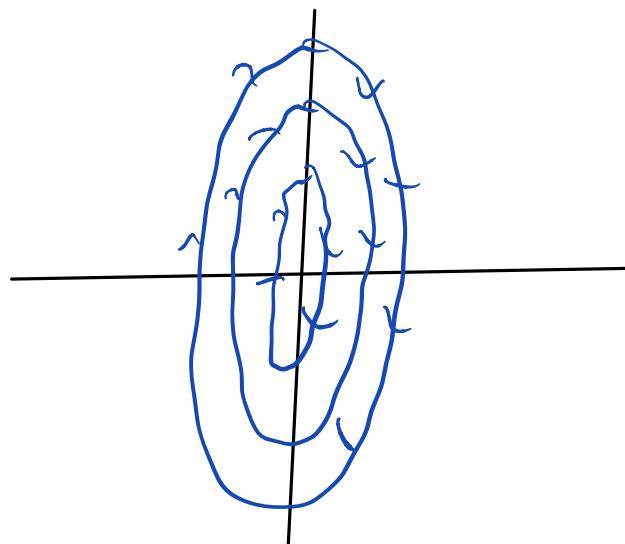
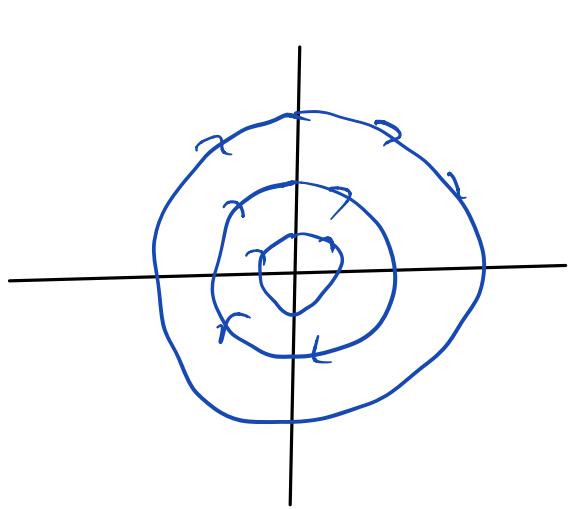
$$\text{Matrix } T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

colonne sono per
rete e immagine
di $(1, 2i)$

$$T^{-1}AT = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = B$$

in forma
canonica

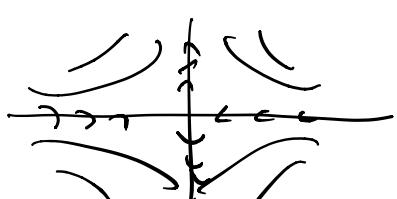


$$y' = By$$

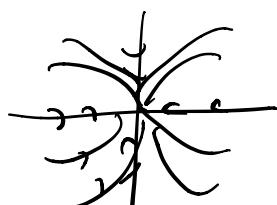
$$x' = Ax$$

Risumendo \rightarrow cosa planare

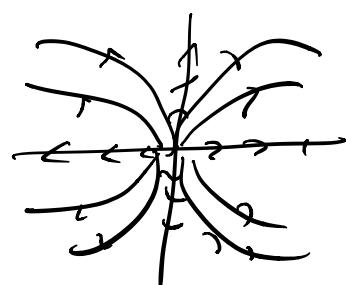
$$\lambda_1 < 0 < \lambda_2$$



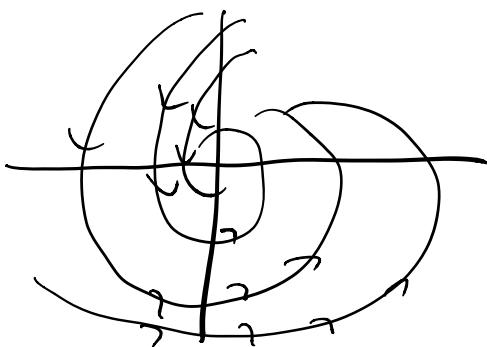
$$\lambda_1 < \lambda_2 < 0$$



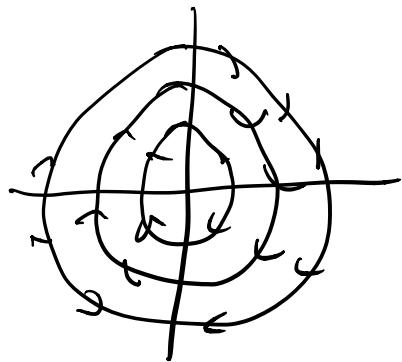
$$0 < \lambda_2 < \lambda_1$$



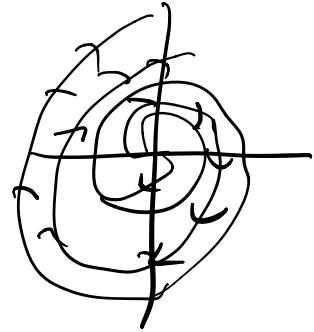
$d \neq i \beta$



$$\alpha < 0$$



$$\alpha = 0$$



$$\alpha > 0$$

Poincaré

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\lambda^2 - \underbrace{(a+d)}_{\text{tr } A} \lambda + \underbrace{(ad - bc)}_{\det A} = 0$$