Lecture 4 : Basic elements of game theory (cont'd)

Best response

The ***Best Response*** of a player is his preferred action given the strategies played by the other players.

- Consider a normal form game with n players
- A strategy s_i is a best response of player i to a given combination of other players' strategies s_{-i} if:

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \text{ for every } s_i' \in S_i$$

- In other words the best response of player i to a given combination of other players' strategies $s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, s_n)$ solves the following problem:

$$\max_{s_i \in S_i} u(s_1, \dots, s_i, \dots, s_n)$$

Note that for each player we can find at least one best response for each combination of other players' strategies.

Example 1

		Player 2		
		L	C	R
Player 1	T	2,3	2,2	5,0
	Y	3,2	5,3	3,1
	Z	4,3	1,1	2,2
	B	1,2	0,1	4,4

The best response of player 1 to player 2 playing L is Z
The best response of player 1 to player 2 playing C is Y
The best response of player 2 to player 1 playing B is R

Example 2

Suppose a game with two players, 1 and 2.

payoff of player 1 is

$$(10 - a_1 - a_2)\, a_1$$

where a_1 is the action of player 1 and a_2 is the action of player 2

What is player 1's best response if player 2 plays $a_2 = 4$?

To find it we have to solve

$$Max\,(10 - a_1 - a_2)a_1$$

The FOC are: $10 - 2a_1 - a_2 = 0$

Solving by a_1 we get player 1'best response

$$a_1 = (10 - a_2)/2$$

Replacing $a_2 = 4$ into player 1'best response we get

$$a_1 = 3$$

Rationalizability

- As with iterated dominance, this procedure eliminates at each stage strategies that are not best response to any strategy combination of the other players.
- The strategies that survive to this process are called ***rationalizable strategies***
- This process requires that players are able to identify best responses. Then respect to iterated dominance we need that player a "more sophisticated".
- To identify a best response could be more difficult respect to identify a strictly dominated strategy
- This ability has to be common knowledge

		Player 2		
		L	C	R
Player 1	T	2,3	2,2	5,0
	Y	3,2	5,3	3,1
	Z	4,3	1,1	2,2
	B	1,2	0,1	4,4

The best response of player 1 to player 2 playing L is Z
The best response of player 1 to player 2 playing C is Y
The best response of player 1 to player 2 playing R is T
Then B is never a best response and we can eliminate it

		Player 2		
		L	C	R
Player 1	T	2,3	2,2	5,0
	Y	3,2	5,3	3,1
	Z	4,3	1,1	2,2
	~~B~~	~~1,2~~	~~0,1~~	~~4,4~~

The best response of player 2 to player 1 playing T is L
The best response of player 2 to player 1 playing Y is C
The best response of player 2 to player 1 playing Z is L
Then R is never a best response and we can eliminate it

		Player 2		
		L	C	~~R~~
Player 1	T	2,3	2,2	~~5,0~~
	Y	3,2	5,3	~~3,1~~
	Z	4,3	1,1	~~2,2~~
	~~B~~	~~1,2~~	~~0,1~~	~~4,4~~

The best response of player 1 to player 2 playing L is Z
The best response of player 1 to player 2 playing C is Y
Then T is never a best response and we can eliminate it

		Player 2		
		L	C	~~R~~
Player 1	~~T~~	~~2,3~~	~~2,2~~	~~5,0~~
	Y	3,2	5,3	~~3,1~~
	Z	4,3	1,1	~~2,2~~
	~~B~~	~~1,2~~	~~0,1~~	~~4,4~~

The best response of player 1 to player 2 playing L is Z
The best response of player 1 to player 2 playing C is Y
Then T is never a best response and we can eliminate it

The process stops here and we have identified the set of rationalizble strategies.

Nash Equilibrium

- It is a prediction about the strategy each player will choose
- This prediction is correct if each player's predicted strategy is a best response to the predicted strategies of the other players.
- Such prediction is *strategically stable* or *self enforcing*: no player wants to change his/her predicted strategy
- We call such a prediction a *Nash Equilibrium*.

Definition of Nash Equilibrium

A strategy profile where all players plays a best response against the strategies of the other players is a **Nash equilibrium**

If for each player there are not indifferences with some other strategies, then we say that the strategy profile is **strict Nash Equilibrium**

Intuitively we can say that a Nash equilibrium strategy profile is a strategy combination where no player has an inventive to deviate from it and she expects that no other player to deviate

Math definition of Nash Equilibrium

The strategy profile $s^* = (s_1^*, \dots, s_n^*)$ is a **Nash equilibrium** if for every player i:

$$u_i(s_1^*, \dots, s_i^*, \dots, s_n^*) \geq u_i\left(s_1^*, \dots, s_i\ \ , \dots, s_n^*\right) \text{ for every action } s_i \in S_i$$

(none ha an incentive to deviate from $s^* = (s_1^*, \dots, s_n^*)$)

The strategy profile $s^* = (s_1^*, \dots, s_n^*)$is a **Strict Nash equilibrium** if for every player i the above inequality holds strictly for every action $s_i \in S_i$

Note

In a Nash equilibrium, for each i, s_i^* solves the following maximization problem:

$$\max_{s_i \in S_i} u_i(s_1^*, \dots, s_i, \dots, s_n^*)$$

Example 1: the Prisoner's Dilemma

		Player 2	
		C(ooperate)	D(efect)
Player 1	C(ooperate)	2,2	$0,\underline{3}$
	D(efect)	$\underline{3},0$	$\underline{1},\underline{1}$

- The unique Nash equilibrium is (D,D)
- For every other profile, at least one player wants to deviate

Example 2: the "Battle of the Sexes"

		Player 2	
		Football	Theatre
Player 1	Football	<u>2</u>,<u>1</u>	0,0
	Theatre	0,0	<u>1</u>,<u>2</u>

- There are two Nash equilibria: (Football, Football) and (Theatre, Theatre)

Example 3: Matching Pennies

		Player 2	
		Head	Tail
Player 1	Head	$\underline{1}$,-1	-1,$\underline{1}$
	Tail	-1,$\underline{1}$	$\underline{1}$,-1

- There is no Nash equilibrium

Example 4: "Stag-Hunt"

		Player 2	
		Stag	Hare
Player 1	Stag	$\underline{2}$,$\underline{2}$	0,1
	Hare	1,0	$\underline{1}$,$\underline{1}$

- There are two equilibria:
- (Stag, Stag) and (Hare, Hare)

Public good game

This is an example of a game with continuous strategies

Two individuals are endowed with 10 pounds.

They can contribute to a public good by delivering any amount of money out of their endowment (c_1, c_2)

For each individual the value of the public good is given by the sum of the contributions multiplied by 0.7

$$v_p = 0.7(c_1 + c_2)$$

Payoff of player 1 = Endowment – contribution + value of the public good

$$\pi_1 = 10 - c_1 + v_p = 10 - c_1 + 0.7(c_1 + c_2)$$

Payoff of player 2 = Endowment – contribution + value of the public good

$$\pi_2 = 10 - c_2 + v_p = 10 - c_2 + 0.7(c_1 + c_2)$$

The best response of player 1 is given by the solution of the problem:

$$\max_{c_1} \pi_1 = 10 - c_1 + 0.7(c_1 + c_2)$$

FOCs are:

$$-1 + 0.7 < 0$$

Then player 1's best response is $c_1 = 0$

The best response of player 2 is given by the solution of the problem:

$$\max_{c_2} \pi_2 = 10 - c_2 + 0.7(c_1 + c_2)$$

FOCs are:

$$-1 + 0.7 < 0$$

Then player 2's best response is $c_2 = 0$

Unique Nash equilibrium: both individuals contribute by 0

But the efficient outcome is both contributing by 10

		Player 2	
		L	R
Player 1	T	3,3	5,4
	M	1,0	4,1
	B	3, 2	1,3

Select the strategy profile that is Nash equilibrium

Mixed strategies: Motivation

		Player 2	
		Head	Tail
Player 1	Head	1,-1	-1,1
	Tail	-1,1	1,-1

- The characteristic of Matching Pennies is that each player wants to outguess the other:
- There are other similar situations where each player wants to outguess the other(s): poker, football, battle,……
- In situations where players want outguess the other, there is no Nash equilibrium **in pure strategies**

Definition of mixed strategy

- A **mixed strategy** of player i is a probability distribution over the strategies in S_i
- The strategies in S_i are called *pure strategies*

  Note: in static games of complete information strategies are the actions the player could take.

Math definition of mixed strategy

Suppose $S_i = \{s_{i1}, \dots s_{ij}, \dots\ s_{iK}\}$ (player *i* has *K* strategies)

A mixed strategy for player *i* is a probability distribution

$$p_i = (p_{i1}, p_{i2}, \dots, p_{iK})$$

where p_{ij} is the probability that player *i* will play strategy *j* (i.e. the probability associated to strategy s_{ij})

Then in a game,

1. $p = (p_1 \dots p_i \dots p_n)$ denotes a ***mixed strategy profile***, i.e. one strategy for each player.

2. p_{-i} denote a mixed strategy profile of all players except player *i*

Example 1: Matching Pennies

- S_i = *{Head, Tail}*
- $(q, 1-q)$ is a mixed strategy where:
  - q is the probability to play *Head* and
  - $1-q$ is the probability to play *Tail* where $0 \leq q \leq 1$
- Note: $(0,1)$ is the pure strategy *Tail and* $(1,0)$ is the pure strategy *Head*

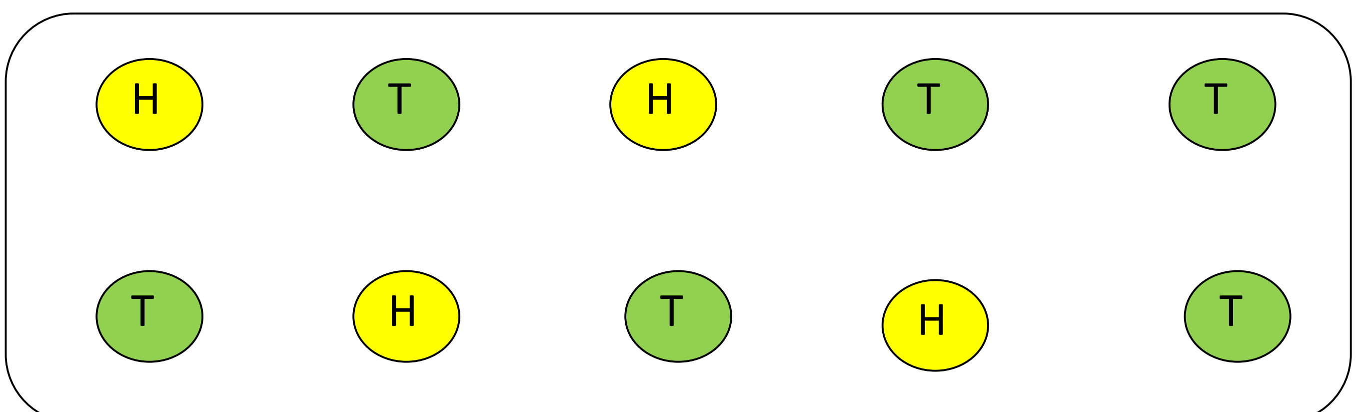
- But what means to play a mixed strategy?

Suppose that Player 1 likes to play:

- Head by probability 0.4
- Tail by probability 0.6

i.e. the mixed strategy $p_1 = (0.4, 0.6)$

The action he will play is randomly chosen according to the distribution $(0.4, 0.6)$, for example choosing a ball from a box where 4 balls are marked by H (Head) and 6 are marked by T (Tail)



		Player 2		
		L	C	R
Player 1	T	2,3	2,2	5,0
	Y	3,2	5,3	3,1
	Z	4,3	1,1	2,2
	B	1,2	0,1	4,4

- $S_2 = \{L, C, R\}$
- $p_2 = (p_{2L}, p_{2C}, p_{2R})$ is a mixed strategy of Player 2 where:
  - $p_{2L} = q$ is the probability to play *L*,
  - $p_{2C} = r$ is the probability to play *C* and
  - $p_{2R} = 1 - q - r$ is the probability to play *R*
- $p_2 = (q, r, 1 - q - r)$
- $0 \leq q \leq 1; 0 \leq r \leq 1; 0 \leq q + r \leq 1$
- Note: $p_2 = (0, 0, 1)$ is the pure strategy *R*

		Player 2		
		L	C	R
Player 1	T	2,3	2,2	5,0
	Y	3,2	5,3	3,1
	Z	4,3	1,1	2,2
	B	1,2	0,1	4,4

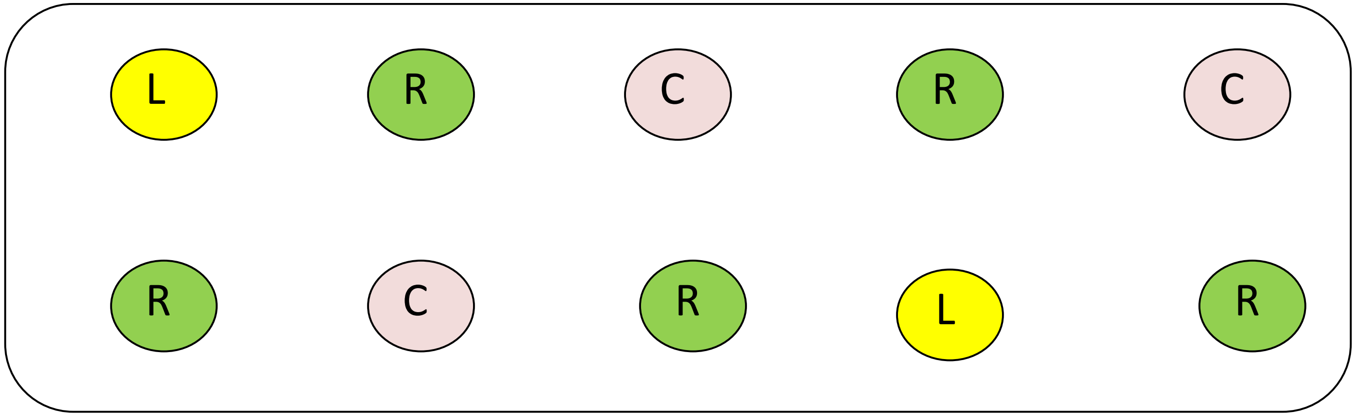
- $S_2 = \{L, C, R\}$
- $p_2 = (p_{2L}, p_{2C}, p_{2R})$ is a mixed strategy of Player 2 where:
  - $p_{2L} = q$ is the probability to play L,
  - $p_{2C} = r$ is the probability to play C and
  - $p_{2R} = 1 - q - r$ is the probability to play R
- $p_2 = (q, r, 1 - q - r)$
- $0 \leq q \leq 1;\ 0 \leq r \leq 1;\ 0 \leq q + r \leq 1$
- Note: $p_2 = (0, 0, 1)$ is the pure strategy R

Suppose that Player 2 wants play:

- L by probability 0.2
- C by probability 0.3
- R by probability 0.5

i.e. the mixed strategy $p_2 = (0.2, 0.3, 0.5)$

The action he will play, it is randomly chosen according to the distribution $(0.2, 0.3, 0.5)$, for example choosing a ball from a box where 2 balls are marked by L, 3 are marked by C and 5 are marked by R



Expected utility and mixed strategies

In order to evaluated a player's strategy against a strategy profile played by others we need to compute its expected value.
For example for player i, the expected utility of strategy p_i against the others' strategies p_{-i} is denoted by $u_i(p_i, p_{-i})$

Example:

		Player 2		
		L	C	R
Player 1	T	2,1	2,2	5,0
	Y	3,2	5,3	3,1
	Z	4,3	1,1	2,2
	B	1,4	0,1	4,4

We compute the expected value of $p_2 = (1, 0, 0)$ (the pure strategy L) against the player 1's strategy $p_1 = (0.4, 0.3, 0.2, 0.1)$

$$E_2(p_2|p_1) = 1 \cdot 0.4 + 2 \cdot 0.3 + 3 \cdot 0.2 + 4 \cdot 0.1 = 2$$

Example:

		Player 2		
		L	C	R
Player 1	T	2,1	2,2	5,3
	Y	3,2	5,3	3,1
	Z	4,3	1,1	2,2
	B	1,4	0,1	4,5

We compute the expected value of $p_2 = (0.6, 0, 0.4)$ (a mixed strategy L) against the player 1's strategy $p_1 = (0.3, 0, 0, 0.7)$

		Player 2		
		0.6	**0**	**0.4**
Player 1	**0.3**	2,1	2,2	5,3
	0	3,2	5,3	3,1
	0	4,3	1,1	2,2
	0.7	1,4	0,1	4,5

$$E(p_2|p_1) = 1 \cdot 0.6 \cdot 0.3 + 3 \cdot 0.4 \cdot 0.3 + 4 \cdot 0.6 \cdot 0.7 + 5 \cdot 0.4 \cdot 0.7$$