1.3 Quantum Dynamics and Complete Positivity

- $\boxed{\text{Schrödinger equation}}$: $i\partial_t|\psi_t\rangle = H|\psi_t\rangle$ $(\hbar=1)$; $H=H^\dagger$: Hamiltonian

Formal solution: $|\psi_t\rangle = e^{-iHt}|\psi\rangle$

$U_t = e^{-iHt}$: unitary dynamics , $U_t^\dagger = U_{-t} = e^{itH}$

Reversible: $U_{-t}U_t = \mathbb{1}$

Group composition law: $U_t \circ U_s = U_{t+s}$ $\forall\, t,s \in \mathbb{R}$

- $\boxed{\text{Liouville-von Neumann equation}}$: $\partial_t\rho_t = -i[H,\rho_t]$

$$\partial_t\left(|\psi_t\rangle\langle\psi_t|\right) = \partial_t|\psi_t\rangle\langle\psi_t| + |\psi_t\rangle\,\partial_t\langle\psi_t| = -i\,H|\psi_t\rangle\langle\psi_t| + i\,|\psi_t\rangle\langle\psi_t|H$$
$$= -i\,[H, |\psi_t\rangle\langle\psi_t|]$$

Formal solution: $|\psi_t\rangle\langle\psi_t| = U_t|\psi\rangle\langle\psi|U_t^\dagger$

initial density matrix: $\rho = \sum_j \lambda_j |\psi_j\rangle\langle\psi_j| \longrightarrow \rho_t = \sum_j \lambda_j\, U_t|\psi_j\rangle\langle\psi_j|U_t^\dagger = U_t\rho U_t^\dagger$

$$\boxed{\partial_t\rho_t = -i\,[H,\rho_t]}$$

Remark: the unitary Schrödinger dynamics defines a linear map $\mathcal{U}_t$ on the space of states: $\boxed{\mathcal{U}_t : \mathcal{S}(S) \ni \rho \longmapsto \rho_t = \mathcal{U}_t[\rho] = U_t \rho U_t^\dagger}$

satisfying the group composition law $\boxed{\mathcal{U}_t \circ \mathcal{U}_s = \mathcal{U}_{t+s} \quad \forall\, s,t \geq 0}$

Definition 1.3.1. (Duality)

Let $B(\mathcal{H})$ the set of bounded linear operators on $\mathcal{H}$, namely

$X \in B(\mathcal{H}) \Longrightarrow \| X|\psi\rangle \| \leq \|X\| \, \|\psi\|$, $\|X\| < +\infty$, $\forall\, |\psi\rangle \in \mathcal{H}$.

$B(\mathcal{H})$ is a so-called C^* algebra: $X^\dagger \in B(\mathcal{H})$, $XY \in B(\mathcal{H})$ if $X, Y \in B(\mathcal{H})$.

To every linear map $\mathcal{L} : \mathcal{S}(S) \to \mathcal{S}(S)$ there corresponds a $\boxed{\text{dual map}}$ $\mathcal{L}^T : B(\mathcal{H}) \to B(\mathcal{H})$ such that $\mathrm{Tr}\left(\mathcal{L}[\rho] X\right) = \mathrm{Tr}\left(\rho\, \mathcal{L}^T[X]\right) \quad \forall \rho \in \mathcal{S}(S), X \in B(\mathcal{H})$

Remark

By duality one moves from the Schrödinger to the Heisenberg picture:
in the first one states change, in the other one operators without altering mean values.

Example 1.3.1

$$\mathcal{S}(S) \ni \rho \xmapsto{\mathcal{U}_t} \rho_t := \mathcal{U}_t[\rho] = U_t \rho U_t^\dagger$$

$$B(\mathcal{H}) \ni X \xmapsto{\mathcal{U}_t^T} X_t := \mathcal{U}_t^T[\rho] = U_t^\dagger \rho U_t$$

Indeed, by cyclicity of trace (property 2 on page 3): $\mathrm{Tr}(\rho_t X) = \mathrm{Tr}(U_t \rho U_t^\dagger X)$

$$= \mathrm{Tr}(\rho U_t^\dagger X U_t) = \mathrm{Tr}(\rho X_t)$$

Irreversible Quantum Dynamics (Measurement Processes)

Given a discrete spectrum observable $A = A^\dagger = \sum_\alpha a_\alpha |a_\alpha\rangle\langle a_\alpha|$ of S

and a state vector $|\psi\rangle \in \mathcal{H}$, measuring A one finds

the eigenvalues a_α with probabilities $|\langle a_\alpha|\psi\rangle|^2$ leaving the system in the state $|a_\alpha\rangle$

If no results are selected the post-measurement state is $\sum_\alpha |\langle a_\alpha|\psi\rangle|^2 |a_\alpha\rangle\langle a_\alpha|$

The process amounts to $|\psi\rangle\langle\psi| \longmapsto \sum_\alpha \langle a_\alpha|\psi\rangle\langle\psi|a_\alpha\rangle |a_\alpha\rangle\langle a_\alpha| = \sum_\alpha P_\alpha |\psi\rangle\langle\psi| P_\alpha$

or on initial mixed states, $\boxed{\rho \longmapsto \sum_\alpha P_\alpha \rho P_\alpha, \quad P_\alpha = |a_\alpha\rangle\langle a_\alpha|}$

$\boxed{\text{Definition 1.3.2.}}$ P.O.V.M. measurement (Positive Operator Valued Measure)

The process $\rho \longmapsto \mathcal{E}_P[\rho] = \sum_\alpha P_\alpha \rho P_\alpha$ with $P_\alpha P_\beta = \delta_{\alpha\beta} P_\alpha$

is called a projective measurement.

In general, given a set $\boxed{X_\alpha \in B(\mathcal{H}) \text{ such that } \sum_\alpha X_\alpha^\dagger X_\alpha = \mathbb{1}}$

the process $\rho \longmapsto \mathcal{E}_X[\rho] = \sum_\alpha X_\alpha \rho X_\alpha^\dagger$

is called a P.O.V.M. measurement.

Remark

The operators X_α need not even be self-adjoint: in any case $\mathcal{E}_X[\rho]$ is again a density matrix: $\mathrm{Tr}\left(\mathcal{E}_X[\rho]\right) = \mathrm{Tr}\left(\left(\sum_\alpha X_\alpha^\dagger X_\alpha\right)\rho\right) = \mathrm{Tr}(\rho) = 1$

$$\langle\psi|\mathcal{E}_X[\rho]\psi\rangle = \sum_\alpha \langle X_\alpha^\dagger\psi|\,\rho|X_\alpha\psi\rangle \geq 0$$

$\boxed{\text{Exercise 1.3.1.}}$: Prove that P.O.V.M.s naturally emerge when using two different Stern-Gerlach apparatuses one after the other.

<u>Remark</u> : $\mathcal{U}_t[\rho] = U_t \rho U_t^\dagger$ and $\mathcal{E}_X[\rho] = \sum_\alpha X_\alpha \rho X_\alpha^\dagger$

are linear maps that preserve both trace and positivity of density matrices.

Exercise 1.3.2. : Prove that the dual map of $\mathcal{E}_X$ (see Definition 1.3.1) is

$$\mathcal{E}_X^T[A] = \sum_\alpha X_\alpha^\dagger A X_\alpha \quad \forall A \in B(\mathcal{H})$$ (use the cyclicity of the trace)

Prove that $\mathcal{E}_X^T$ <u>preserves positivity</u> but <u>not necessarily</u> trace.

Definition 1.3.3. Positivity preserving linear maps on $\mathcal{B}(\mathcal{H})$ are called **positive**

$\mathcal{E} : B(\mathcal{H}) \to B(\mathcal{H})$ such that $\mathcal{E}[X] \geq 0 \quad \forall X \geq 0$ is called positive

<u>Remark</u> : $\mathcal{E}$ above acts on operators $\mathrm{Tr}(\rho \mathcal{E}[X]) = \mathrm{Tr}(\mathcal{E}'[\rho] X)$ defines its dual action $\mathcal{E}'$ on states.

– $\boxed{\text{Coupling to ancillas}}$

Every quantum system S described by a Hilbert space $\mathcal{H}$ can be statistically coupled to a finite n-level ancillary quantum system S_n described by the Hilbert space $\mathbb{C}^n$. The Hilbert space of the compound system is $\mathcal{H} \otimes \mathbb{C}^n$ with state vectors of the form

$$|\psi\rangle \otimes \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{pmatrix} = \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \\ \vdots \\ |\psi_n\rangle \end{pmatrix} \qquad |\psi\rangle, |\psi_i\rangle = \phi_i |\psi\rangle \in \mathcal{H}$$

and their linear combinations.

The compound observables form the algebra $B(\mathcal{H}) \otimes M_n(\mathbb{C}) = M_n(B(\mathcal{H}))$ when $M_n(\mathbb{C})$ is the algebra of $n \times n$ complex matrices, the observables of S_n, and $M_n(B(\mathcal{H}))$ denotes the $n \times n$ matrices with entries from $B(\mathcal{H})$, of the form

$$X \otimes \sum_{i,j=1}^{n} y_{ij} |i\rangle\langle j| = \sum_{i,j=1}^{n} X_{ij} \otimes |i\rangle\langle j| = \begin{pmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & & \\ X_{n1} & \cdots & X_{nn} \end{pmatrix} \qquad X_{ij} = y_{ij} X \in B(\mathcal{H})$$

and their linear combinations.

$\boxed{\text{Exercise 1.3.3.}}$ Write down the explicit action of a generic $X \in B(\mathcal{H}) \otimes M_n(\mathbb{C})$ on a generic $|\psi\rangle \in \mathcal{H} \otimes \mathbb{C}^n$.

$\{|i\rangle\}_{i=1}^n$ ONB in $\mathbb{C}^n$; $\{|i\rangle\langle j|\}_{i,j=1}^n$ matrix units in $M_n(\mathbb{C})$

$X_{ij} \in B(\mathcal{H})$ $\qquad |\psi_k\rangle \in \mathcal{H}$

$$X = \sum_{i,j=1}^n X_{ij} \otimes |i\rangle\langle j| \,, \quad |\psi\rangle = \sum_{k=1}^n |\psi_k\rangle \otimes |k\rangle$$

$$\boxed{X|\psi\rangle = \sum_{i,j,k=1}^n X_{ij}|\psi_k\rangle \otimes |i\rangle\langle j|k\rangle = \sum_{i,j=1}^n X_{ij}|\psi_j\rangle \otimes |i\rangle}$$

$\boxed{\text{Definition 1.3.4.}}$ The lifting of a linear map $\mathcal{L} : B(\mathcal{H}) \to B(\mathcal{H})$ to $B(\mathcal{H}) \otimes M_n(\mathbb{C})$ is $\boxed{\mathcal{L} \otimes \mathrm{id}_n}$ where $\boxed{\mathrm{id}_n[|i\rangle\langle j|] = |i\rangle\langle j|}$ for all matrix units in $M_n(\mathbb{C})$ denotes the identity map

$\boxed{\text{Question}}$: if $\mathcal{L} : B(\mathcal{H}) \to B(\mathcal{H})$ is positive

is it such also $\mathcal{L} \otimes \mathrm{id}_n : B(\mathcal{H}) \otimes M_n(\mathbb{C}) \circlearrowleft$?

$\boxed{\text{Answer}}$: NO

$\boxed{\text{Example 1.3.2.}}$ Consider the $\boxed{\text{transposition}}$ on $M_2(\mathbb{C})$.

$$\tau\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

It is a positive linear map since it does not change the spectrum of matrices. Consider the action of $\tau \otimes \mathrm{id}_2$ on the positive operator $P_{00} = |\Psi_{00}\rangle\langle\Psi_{00}|$ (see Example 1.2.1)

$$\tau \otimes \mathrm{id}_2 [P_{00}] = \frac{1}{4}\left[1 \otimes 1 + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z\right] = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Therefore $P_{00} \geq 0$ is turned by $\tau \otimes \mathrm{id}_2$ into an operator with a negative eigenvalue.

$\boxed{\tau}$ is $\boxed{\text{positive}}$ but $\boxed{\tau \otimes \mathrm{id}_2}$ is $\boxed{\text{NOT positive}}$

– Positive maps $\mathcal{L} : B(\mathcal{H}) \circlearrowleft$ that remain positive when lifted are called <u>Completely Positive</u> (CP)

$\boxed{\text{Definition 1.3.5.}}$

A linear map $\mathcal{L} : B(\mathcal{H}) \circlearrowleft$ is called $\boxed{\text{n-positive}}$ if $\boxed{\mathcal{L} \otimes id_n} : B(\mathcal{H}) \otimes M_n(\mathbb{C}) \circlearrowleft$ is $\boxed{\text{positive}}$;

$\boxed{\text{completely positive}}$ if $\mathcal{L}$ is $\boxed{\text{n-positive}}$ $\boxed{\text{for all } n \geq 1}$

<u>Remark</u> : when $n = 1$ $M_n(\mathbb{C}) = \mathbb{C}$ and $\mathcal{L} \otimes id_1 = \mathcal{L}$

so that 1-positivity = positivity

$\boxed{\text{QUESTION}}$: How does one identify CP maps?

Remark : transposition $\mathcal{T}$ maps qubit states into qubit states

$\mathcal{T}\otimes id_2$ maps the entangled state $|\psi_0\rangle\langle\psi_0|$ into $\mathcal{T}\otimes id_2[|\psi_0\rangle\langle\psi_0|]$ which has negative eigenvalues and cannot be a bona fide state of the compound system $S_2 + S_2$.

Remark : Lack of positivity preservation occurs only with $\boxed{\text{entangled states}}$. If all states of $S + S_n$ were $\boxed{\text{separable}}$ (see Definition 1.2.4.)

Then

$$\rho = \sum_{i,j} d_{ij}\, \rho_{1i}\otimes\rho_{2j} \longrightarrow \boxed{\mathcal{L}\otimes id_n[\rho]} = \sum_{i,j} \underbrace{d_{ij}}_{\geq 0}\, \underbrace{\mathcal{L}[\rho_{1i}]}_{\geq 0}\otimes \underbrace{\rho_{2j}}_{\geq 0} \boxed{\geq 0}$$

$\boxed{\text{for all positive}}$ linear maps $\mathcal{L} : B(\mathcal{H}) \to B(\mathcal{H})$.

– Step 1 : identify positive operators in $B(\mathcal{H}) \otimes M_n(\mathbb{C})$

$\boxed{\text{Lemma 1.3.1.}}$ Elementary operators of the form

$$X_{ee} := \sum_{i,j=1}^{n} X_i^{\dagger} X_j \otimes |i\rangle\langle j| \in B(\mathcal{H}) \otimes M_n(\mathbb{C})$$

with $\{X_i\}_{i=1}^{n} \in B(\mathcal{H})$, $\{|i\rangle\}_{i=1}^{n}$ ONB in $\mathbb{C}^n$

are positive.

$\boxed{\text{Proof}}$ For generic state vectors $|\psi\rangle = \sum_{k=1}^{n} |\psi_k\rangle \otimes |k\rangle \in \mathcal{H} \otimes \mathbb{C}^n$

$$\langle \psi | X_{ee} \psi \rangle = \sum_{i,j=1}^{n} \langle \psi_i | X_i^{\dagger} X_j \psi_j \rangle = \Big\| \sum_{j=1}^{n} X_j |\psi_j\rangle \Big\|^2 \geq 0$$

<u>Remark</u> : Linear combinations of X_{ee} with positive coefficients are also positive:

$$X = \sum_{\alpha} d_{\alpha} X_{ee}^{\alpha} \geq 0 \Leftarrow d_{\alpha} \geq 0$$

— From now on we shall consider both S and S_n finite level systems, $S = S_m$, so that $S_m + S_n$ is described by the Hilbert space $\mathbb{C}^m \otimes \mathbb{C}^n = \mathbb{C}^{mn}$ and by the algebra $M_m(\mathbb{C}) \otimes M_n(\mathbb{C}) = M_n(M_m(\mathbb{C}))$

$\boxed{\text{Lemma 1.3.2}}$ Positive $X \in M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$ are combinations of $X_{ee} \geq 0$ with positive coefficients.

$\boxed{\text{Proof.}}$ From spectralization $0 \leq X = \sum_{\alpha=1}^{mn} x_\alpha |\phi_\alpha\rangle\langle\phi_\alpha|$ with eigenvalues $x_\alpha \geq 0$ and eigenvectors $\langle\phi_\alpha|\phi_\beta\rangle = \delta_{\alpha\beta}$

Writing $|\phi_\alpha\rangle = \sum_{k=1}^{n} |\phi_{\alpha k}\rangle \otimes |k\rangle$, $|\phi_{\alpha k}\rangle \in \mathbb{C}^m$, $\{|k\rangle\}_{k=1}^{n}$ ONB in $\mathbb{C}^n$

one gets
$$X = \sum_{\alpha=1}^{mn} x_\alpha \sum_{k,\ell=1}^{n} |\phi_{\alpha k}\rangle\langle\phi_{\alpha\ell}| \otimes |k\rangle\langle\ell|$$
$$= \sum_{\alpha=1}^{mn} x_\alpha X^\alpha_{e\ell} \quad ; \quad X^\alpha_{ee} := \sum_{k,\ell=1}^{n} \left(|\phi_{\alpha k}\rangle\langle\phi|\right)\left(|\phi\rangle\langle\phi_{\alpha\ell}|\right) \otimes |k\rangle\langle\ell|$$

that is $X^\alpha_{ee} = \sum_{k,\ell=1}^{n} X^\dagger_{\alpha k} X_{\alpha\ell} \otimes |k\rangle\langle\ell|$ with $X_{\alpha\ell} = |\phi\rangle\langle\phi_{\alpha\ell}|$

$\forall\ |\phi\rangle \in \mathbb{C}^m$ such that $\langle\phi|\phi\rangle = 1$

<u>Remark</u> : Lemma 1.3.2 says that n-positivity can be checked on positive operators X_{el}.

$\boxed{\text{Proposition 1.3.1.}}$ $\mathcal{L} : M_m(\mathbb{C}) \longrightarrow M_m(\mathbb{C})$ is CP $\boxed{\text{if}}$

$$M_m(\mathbb{C}) \ni X \longmapsto \mathcal{L}[X] = \sum_\alpha L_\alpha^\dagger X L_\alpha \quad , \quad L_\alpha \in M_m(\mathbb{C}).$$

$\boxed{\text{Proof.}}$

$$\mathcal{L} \otimes \mathrm{id}_n [X_{el}] = \sum_{i,j=1}^{n} \mathcal{L}[X_i^\dagger X_j] \otimes |i\rangle\langle j|$$

$$= \sum_\alpha \sum_{i,j=1}^{n} \underbrace{(X_i L_\alpha)^\dagger}_{Y_{i\alpha}^\dagger} \underbrace{(X_j L_\alpha)}_{Y_{j\alpha}} \otimes |i\rangle\langle j| \geq 0$$

Indeed, $\mathcal{L} \otimes \mathrm{id}_n[X_{el}]$ is a positive elementary generator for all $n \geq 1$.

$\boxed{\text{Example 1.3.3.}}$ The unitary dynamics $X \mapsto X_t = U_t^\dagger X U_t$

and the P.O.V.M. processes $X \mapsto \sum_\alpha X_\alpha^\dagger X X_\alpha$

are CP (see Remark on page 23 and Exercise 1.3.2.)

- Step 2: Show that all CP maps $\mathcal{L} : M_m(\mathbb{C}) \circlearrowleft$ are of the form

$$\mathcal{L} : M_m(\mathbb{C}) \ni X \longmapsto \mathcal{L}[X] = \sum_\alpha L_\alpha^\dagger X L_\alpha$$

$\boxed{\text{Theorem 1.3.1.}}$ $\mathcal{L} : M_m(\mathbb{C}) \circlearrowleft$ is CP if and only if

$\boxed{\text{Choi Matrix of } \mathcal{L}}$ $M_{\mathcal{L}} := \mathcal{L} \otimes \mathrm{id}_m [P^{(m)}_{unif}] \geq 0$ where $P^{(m)}_{unif} = |\Psi^{(m)}_{unif}\rangle\langle\Psi^{(m)}_{unif}|$

$$|\Psi^{(m)}_{unif}\rangle = \frac{1}{\sqrt{m}} \sum_{j=1}^m |j\rangle \otimes |j\rangle$$ (see Exercise 1.2.2.)

$\boxed{\text{Proof}}$

If $\mathcal{L}$ is CP, then $\mathcal{L} \otimes \mathrm{id}_n$ is positive for all n, thus also for $n = m$ whence $\mathcal{L} \otimes \mathrm{id}_m$ must preserve the positivity of $P^{(m)}_{unif}$.

If $X = \mathcal{L} \otimes \mathrm{id}_m [P^{(m)}_{unif}] \geq 0$ then by spectralization

$$X = \sum_{\alpha=1}^{m^2} x_\alpha |\Psi_\alpha\rangle\langle\Psi_\alpha| \qquad x_\alpha \geq 0 \qquad \langle\Psi_\alpha|\Psi_\beta\rangle = \delta_{\alpha\beta}$$

$$|\Psi_\alpha\rangle = \sum_{j=1}^m |\Psi_{\alpha j}\rangle \otimes |j\rangle = \sum_{j=1}^m \left(V_\alpha^\dagger |j\rangle\right) \otimes |j\rangle = \tilde{V}_\alpha^\dagger \otimes \mathbb{1} |\Psi^{(m)}_{unif}\rangle$$

where $\langle k|\tilde{V}_\alpha^\dagger|j\rangle = \sqrt{m}\,\langle k|\Psi_{\alpha j}\rangle$. Then,

$$\boxed{\mathcal{L} \otimes \mathrm{id}_m}[P^{(m)}_{unif}] = \sum_\alpha \boxed{L_\alpha^\dagger \otimes \mathbb{1}}\, P^{(m)}_{unif}\, \boxed{L_\alpha \otimes \mathbb{1}}, \quad \boxed{L_\alpha = \sqrt{x_\alpha}\,\tilde{V}_\alpha}$$

The proof is concluded upon using the next lemma.

$\boxed{\text{Lemma 1.3.3.}}$ $\mathcal{L}\otimes id_m\left[P^{(m)}_{unif}\right] = \sum_\alpha L^\dagger_\alpha \otimes 1\, P^{(m)}_{unif}\, L_\alpha \otimes 1$

iff $\mathcal{L}[X] = \sum_\alpha L^\dagger_\alpha\, X\, L_\alpha \quad \forall\, X \in M_m(\mathbb{C})$.

$\boxed{\text{Proof}}$ $\mathcal{L}$ is determined by its action $\boxed{\mathcal{L}[|i\rangle\langle j|]}$ on the matrix units;

indeed, $\mathcal{L}[X] = \sum_{i,j=1}^m X_{ij}\, \mathcal{L}[|i\rangle\langle j|]$ for $X = \sum_{i,j=1}^m X_{ij}\, |i\rangle\langle j|$.

$\mathcal{L}[|i\rangle\langle j|]$ is in turn defined by the matrix entries $\boxed{\langle k|\mathcal{L}[|i\rangle\langle j|]|e\rangle}$.

These follows from those of $\mathcal{L}\otimes id_m\left[P^{(m)}_{unif}\right]$

$$m\,\langle k\otimes i|\,\mathcal{L}\otimes id_m\left[P^{(m)}_{unif}\right]|e\otimes j\rangle = \sum_{a,b=1}^m \langle k|\mathcal{L}[|a\rangle\langle b|]|e\rangle\langle i|a\rangle\langle b|j\rangle$$
$$= \langle k|\,\mathcal{L}[|i\rangle\langle j|]|e\rangle$$

Remark : The complete positivity of $\mathcal{L}: M_m(\mathbb{C}) \circlearrowleft$ is checked not on all $\mathcal{L}\otimes id_n$, $n=1,2,\ldots$, but only for $n=m$ and looking not at all $\mathcal{L}\otimes id_m[X]$ for all positive $X \in M_m(\mathbb{C})\otimes M_m(\mathbb{C})$ but only for $X = P^{(m)}_{unif}$, the uniform projection.

(see Example 1.3.2.)

$\boxed{\text{Definition 1.3.6.}}$ A CP map $\mathcal{L} : M_m(\mathbb{C}) \circlearrowleft$ is called unital (CPU) iff $\mathcal{L}[\mathbb{1}] = \sum_\alpha L_\alpha^\dagger L_\alpha = \mathbb{1}$.

$\boxed{\text{Lemma 1.3.4.}}$ $\mathcal{L} : M_m(\mathbb{C}) \circlearrowleft$ is CPU iff its dual $\mathcal{L}'$ on $\mathcal{S}(M_m(\mathbb{C}))$ (space of states) preserves the trace.

$\boxed{\text{Proof}}$ $\mathcal{L}[X] = \sum_\alpha L_\alpha^\dagger X L_\alpha$, $\quad \mathcal{L}'[\rho] = \sum_\alpha L_\alpha \rho L_\alpha^\dagger$

$X \in M_m(\mathbb{C}) \qquad\qquad \rho \in \mathcal{S}(M_m(\mathbb{C}))$

(With the notation of Definition 1.3.1.: $(\mathcal{L}')^T = \mathcal{L}$)

$$\sum_\alpha L_\alpha^\dagger L_\alpha = \mathbb{1} \Rightarrow \mathrm{Tr}\,\rho = \mathrm{Tr}(\rho \mathbb{1}) = \mathrm{Tr}\left(\sum_\alpha L_\alpha \rho L_\alpha^\dagger\right) = \mathrm{Tr}(\mathcal{L}'[\rho])$$

$$\mathrm{Tr}(\mathcal{L}'[\rho]) = \mathrm{Tr}\,\rho = \mathrm{Tr}\left(\sum_\alpha L_\alpha \rho L_\alpha^\dagger\right) = \mathrm{Tr}\left(\rho \sum_\alpha L_\alpha^\dagger L_\alpha\right) \quad \forall \rho \in \mathcal{S}(M_m(\mathbb{C}))$$

implies $\sum_\alpha L_\alpha^\dagger L_\alpha = \mathbb{1}$ (convince yourselves).

$\boxed{\text{Exercise 1.3.4.}}$ Transposition on $M_n(\mathbb{C})$: positive map

$$\tau \otimes id_n \left[P^{(n)}_{unif} \right] = \frac{1}{n} \sum_{i,j=1}^{n} \tau\left[|i\rangle\langle j| \right] \otimes |i\rangle\langle j| = \frac{1}{n} \sum_{i,j=1}^{n} |j\rangle\langle i| \otimes |i\rangle\langle j|$$

$\boxed{V := \sum_{i,j=1}^{n} |j\rangle\langle i| \otimes |i\rangle\langle j|}$ is the so-called $\boxed{\text{Flip operator}}$

$$V |\psi \otimes \phi\rangle = \sum_{i,j=1}^{n} \langle i|\psi\rangle \langle j|\phi\rangle \; |j\rangle \otimes |i\rangle = |\phi \otimes \psi\rangle$$

$$V^2 = \mathbb{1} \; ; \quad V \frac{|ij\rangle + |ji\rangle}{\sqrt{2}} = \frac{|ij\rangle + |ji\rangle}{\sqrt{2}} \, , \quad V|ii\rangle = |ii\rangle \, , \quad V \frac{|ij\rangle - |ji\rangle}{\sqrt{2}} = - \frac{|ij\rangle - |ji\rangle}{\sqrt{2}}$$

$\tau \otimes id_n \left[P^{(n)}_{unif} \right]$ has eigenvalue -1 with multiplicity $\frac{n(n-1)}{2}$

$\boxed{\text{Definition 1.3.7.}}$ Given a linear map $\mathcal{L} : M_n(\mathbb{C}) \circlearrowleft$

$\mathcal{L} \otimes id_n \left[P^{(n)}_{unif} \right]$ is called the Choi-matrix of $\mathcal{L}$.

$\mathcal{L}$ is CP iff its Choi-matrix is positive semi-definite.

$\boxed{\text{Example 1.3.4.}}$ Transposition on qubits

Pauli matrix algebra : $\sigma_i\sigma_j = i\,\varepsilon_{ijk}\sigma_k$ $\begin{cases} \sigma_1\sigma_2 = i\sigma_3 \\ \sigma_2\sigma_3 = i\sigma_1 \\ \sigma_3\sigma_1 = i\sigma_2 \end{cases}$

$\sigma_i^2 = \mathbb{1}$

$$\boxed{i \neq j \Rightarrow \sigma_i\sigma_j\sigma_i = -\sigma_j}$$

Consider the map $\mathcal{L}[X] = \frac{1}{2}\left(X + \sigma_x X \sigma_x - \sigma_y X \sigma_y + \sigma_z X \sigma_z\right) \quad \forall X \in M_2(\mathbb{C})$

$\mathcal{L}[\mathbb{1}] = \mathbb{1}$, $\mathcal{L}[\sigma_x] = \sigma_x$, $\mathcal{L}[\sigma_y] = -\sigma_y$ $\mathcal{L}[\sigma_z] = \sigma_z$

It acts exactly as the transposition on the Pauli matrices.

$\boxed{\text{Exercise 1.3.5}}$ Hilbert-Schmidt scalar product

Prove that $(X,Y) \longmapsto \langle\langle X, Y\rangle\rangle := \mathrm{Tr}(X^\dagger Y)$ $X, Y \in M_n(\mathbb{C})$

has all the properties of a scalar product.

$\boxed{\text{Example 1.3.5}}$ The normalized Pauli matrices $\left\{\frac{\sigma_0}{\sqrt{2}}, \frac{\sigma_1}{\sqrt{2}}, \frac{\sigma_2}{\sqrt{2}}, \frac{\sigma_3}{\sqrt{2}}\right\} =: \{\tilde{\sigma}_\mu\}_{\mu=0}^3$

form a Hilbert-Schmidt ONB in $M_2(\mathbb{C})$.

$\langle\langle \tilde{\sigma}_\mu, \tilde{\sigma}_\nu \rangle\rangle = \delta_{\mu\nu}$; $M_2(\mathbb{C}) \ni X = \sum_{\mu=0}^{3} X_\mu \tilde{\sigma}_\mu$, $X_\mu = \langle\langle \tilde{\sigma}_\mu, X\rangle\rangle = \mathrm{Tr}\left(\frac{X\sigma_\mu}{\sqrt{2}}\right)$

$\boxed{\text{Example 1.3.6}}$ The matrix units $|i\rangle\langle j| =: e_{ij} \in M_n(\mathbb{C})$ form a HS orthonormal basis.

$$\langle\langle e_{ij} | e_{ke} \rangle\rangle = \mathrm{Tr}(e_{ij}^{\dagger} e_{ke}) = \mathrm{Tr}(|j\rangle\langle i| k\rangle\langle e|) = \langle i|k\rangle\langle e|j\rangle = \delta_{ik}\delta_{je}$$

$$M_n(\mathbb{C}) \ni X = \sum_{i,j=1}^{n} x_{ij} |i\rangle\langle j|$$

$\boxed{\text{Exercise 1.3.6}}$ Construct a HS ONB in $M_n(\mathbb{C})$ consisting of self-adjoint matrices.

$$f_{ij}^{(1)} = \frac{e_{ij} + e_{ji}}{2}, \quad f_{ij}^{(2)} = \frac{e_{ij} - e_{ji}}{2i} \quad i \neq j \quad ; \quad f_{ii}^{(3)} = |i\rangle\langle i|$$

<u>Remark</u> self-adjoint $X = X^{\dagger} \in M_n(\mathbb{C})$ can be represented by real vectors $|x\rangle \in \mathbb{R}^{n^2}$:

$$X = \sum_{i,j=1}^{n} x_{ij}^{(1)} f_{ij}^{(1)} + \sum_{i,j=1}^{n} x_{ij}^{(2)} f_{ij}^{(2)} + \sum_{i=1}^{n} x_{ii}^{(3)} f_{ii}^{(3)}$$

$$X^{\dagger} \stackrel{!}{=} \sum_{i,j=1}^{n} \overline{x_{ij}^{(1)}} f_{ij}^{(1)} + \sum_{i,j=1}^{n} \overline{x_{ij}^{(2)}} f_{ij}^{(2)} + \sum_{i=1}^{n} \overline{x_{ii}^{(3)}} f_{ii}^{(3)}$$

$$\langle\langle X | Y \rangle\rangle = \mathrm{Tr}(X^{\dagger} Y) = \mathrm{Tr}(XY)$$

$$\stackrel{!}{=} \sum_{i,j=1}^{n} x_{ij}^{(1)} y_{ij}^{(1)} + \sum_{i,j=1}^{n} x_{ij}^{(2)} y_{ij}^{(2)} + \sum_{i=1}^{n} x_{ii}^{(3)} y_{ii}^{(3)} = \langle x | y \rangle$$

<u>Remark</u>: Positive maps are not fully physically consistent unless they are also Completely Positive.
If $\Lambda : M_d(\mathbb{C}) \circlearrowleft$ is only Positive then the entangled state $P^{(d)}_{unif} \in \mathcal{S}(S+S)$ does not stay positive under $\Lambda \otimes id$.
In a few words: Λ can be good for one quantum system S but $\Lambda \otimes id$ need not be good for $S+S$ unless Λ is Completely Positive.

Exercise 1.3.7

Transposition is not a physically consistent map. Try a physical explanation of this fact.
T is in quantum mechanics associated with "time-inversion": it must be generated globally, it cannot be performed just on parts of a system as $T \otimes id$ does.