$\boxed{\text{Example 1.3.7}}$ qubit irreversible (semigroup) dynamics

$$\rho = \frac{1+\vec{r}\cdot\vec{\sigma}}{2} = \frac{1}{2}\begin{pmatrix} 1+r_3 & r_1 - i r_2 \\ r_1 + i r_2 & 1 - r_3 \end{pmatrix} \xrightarrow[t\geq 0]{\gamma_t} \rho_t = \frac{1+\vec{r}_t\cdot\vec{\sigma}}{2} = \frac{1}{2}\begin{pmatrix} 1+r_3 e^{-at} & (r_1 - i r_2)e^{-bt} \\ (r_1+ir_2)e^{-bt} & 1-r_3 e^{-at}\end{pmatrix} \quad (1)$$

$\mathrm{Tr}\,\rho = 1$; $\mathrm{Det}\,\rho = \frac{1}{4}\left(1-(r_1^2+r_2^2+r_3^2)\right) \geq 0$ $\qquad$ $\mathrm{Tr}\,\rho_t = 1$; $\mathrm{Det}\,\rho_t = \frac{1}{4}\left(1-(r_1^2 e^{-2bt}+r_2^2 e^{-2bt}+r_3^2 e^{-2at})\right)$

$$\boxed{a,b>0} \Longrightarrow \begin{cases} \rho_t \xrightarrow[t\to+\infty]{} \rho_\infty = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} \\ \mathrm{Det}\,\rho_t \geq \mathrm{Det}\,\rho_t \geq 0 \Longrightarrow \boxed{\gamma_t \ \underline{\text{Positive}}, \text{ Linear map}} \end{cases}$$

- Does γ_t posses a generator L such that $\gamma_t = e^{tL}$ so that $\partial_t \rho_t = L[\rho_t]$ and $\{\gamma_t\}_{t\geq 0}$ form a semigroup: $\gamma_t \circ \gamma_s = \gamma_{t+s}$ $\forall s,t \geq 0$?

From (1) : $\rho_t = \frac{1}{2}\left(1 + r_1 e^{-bt}\sigma_1 + r_2 e^{-bt} + r_3 e^{-at}\right) = \frac{1}{2}\left(1 + r_1\gamma_t[\sigma_1] + r_2\gamma_t[\sigma_2] + r_3\gamma_t[\sigma_3]\right)$

$$\left.\begin{aligned} &\gamma_t[\sigma_1] = e^{-bt}\sigma_1 \ ; \quad \gamma_t[\sigma_2] = e^{-bt}\sigma_2 \ ; \quad \gamma_t[\sigma_3] = e^{-at}\sigma_3 \\ &L[\sigma_1] = -b\,\sigma_1 \ ; \quad L[\sigma_2] = -b\,\sigma_2 \ ; \quad L[\sigma_3] = -a\,\sigma_3 \end{aligned}\right\} \quad (2)$$

- Trace-preservation : $\mathrm{Tr}\, L[\rho] = 0$
- Pauli-algebra : $\sigma_i \sigma_j \sigma_i = -\sigma_j \quad i \neq j$

$$L[\rho] = \alpha\left(\sigma_1 \rho \sigma_1 - \rho\right) + \beta\left(\sigma_2 \rho \sigma_2 - \rho\right) + \gamma\left(\sigma_3 \rho \sigma_3 - \rho\right) \qquad (3)$$

$$\left.\begin{aligned} L[\sigma_1] &= -2(\beta+\gamma) = -b \\ L[\sigma_2] &= -2(\alpha+\gamma) = -b \\ L[\sigma_3] &= -2(\alpha+\beta) = -a \end{aligned}\right\} \Longrightarrow \boxed{\begin{aligned} \beta+\gamma &= \tfrac{b}{2} \\ \alpha+\gamma &= \tfrac{b}{2} \\ \alpha+\beta &= \tfrac{a}{2} \end{aligned}} \Longrightarrow \boxed{\begin{aligned} \alpha = \beta &= \tfrac{a}{4} \\ \gamma &= \tfrac{2b-a}{4} \end{aligned}}$$

$$L[\rho] = \frac{a}{4}\left(\sigma_1 \rho \sigma_1 - \rho\right) + \frac{a}{4}\left(\sigma_2 \rho \sigma_2 - \rho\right) + \frac{2b-a}{4}\left(\sigma_3 \rho \sigma_3 - \rho\right) \qquad (4)$$

$$= \frac{a}{4}\left(\sigma_1 \rho \sigma_1 + \sigma_2 \rho \sigma_2\right) + \frac{2b-a}{4}\, \sigma_3 \rho \sigma_3 - \frac{2b+a}{4}\, \rho \qquad (5)$$

- Notice : $L[1] = \frac{a}{4}(1+1) + \frac{2b-a}{4} - \frac{2b+a}{4} = \frac{a}{2} - \frac{a}{2} = 0$

$$\boxed{\text{Is } \gamma_t \text{ fully physically consistent ?}}$$

Only if γ_t is not only Positive but also Completely Positive.

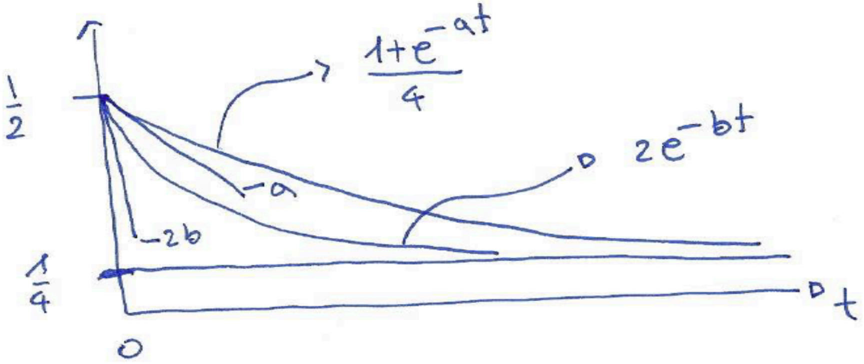
That is if and only if the Choi matrix $\boxed{M_{\gamma_t} := \gamma_t \otimes id\left[P^{(2)}_{unif}\right] \geq 0}$

- Use $P^{(2)}_{unif} = \frac{1}{4}\left(1 \otimes 1 + \sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3\right)$

$$M_{\gamma_t} = \frac{1}{4}\left(1 \otimes 1 + e^{-bt}\left(\sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2\right) + e^{-at}\,\sigma_3 \otimes \sigma_3\right)$$

$$\overset{!}{=} \frac{1}{4}\begin{pmatrix} 1+e^{-at} & 0 & 0 & 2e^{-bt} \\ 0 & 1-e^{-at} & 0 & 0 \\ 0 & 0 & 1-e^{-at} & 0 \\ 2e^{-bt} & 0 & 0 & 1+e^{-at} \end{pmatrix}; \quad \mathrm{Eig}\left[M_{\gamma_t}\right] = \begin{cases} (1-e^{-at})/4 \\ (1-e^{-at})/4 \\ (1+e^{-at}+2e^{-bt})/4 \\ (1+e^{-at}-2e^{-bt})/4 \end{cases}$$

$$\frac{1+e^{-at}-2e^{-bt}}{4} \geq 0 \iff \boxed{a \leq 2b}$$



$a \geq 0$, $b \geq 0$: necessary for Positivity of γ_t

$a \geq 0$, $b \geq 0$, $a \leq 2b$: necessary for Complete Positivity of γ_t

$$L[\rho] = \frac{a}{4}\left(\sigma_1 \rho \sigma_1 - \frac{1}{2}\{\sigma_1^2, \rho\}\right) \qquad \frac{a}{4} \geq 0$$

$$+ \frac{a}{4}\left(\sigma_2 \rho \sigma_2 - \frac{1}{2}\{\sigma_2^2, \rho\}\right) \qquad \frac{a}{4} \geq 0$$

$$+ \frac{2b-a}{4}\left(\sigma_3 \rho \sigma_3 - \frac{1}{2}\{\sigma_3^2, \rho\}\right) \qquad \frac{2b-a}{4} \geq 0$$

$$\boxed{\{A,B\} := \cancel{T}AB + BA}$$

<u>Remark</u> : Entanglement $\Longrightarrow$ Complete Positivity

Complete Positivity $\Longrightarrow$ $2b - a \geq 0 \iff a \leq 2b$

The existence of non-local correlations forces a hyerarchy on the decay times.

<u>Remark</u> : the previous example illustrates the following general result:

$\boxed{\text{Theorem 1.3.2}}$ Gorini-Kossakowski-Sudarshan-Lindblad

A trace-preserving semigroup $\gamma_t : \mathcal{S}(S) \to \mathcal{S}(S)$ on the states of a d-level quantum system S consist of Completely Positive maps γ_t iff

$\gamma_t = e^{tL}$ where

$$L[\rho] = -i[H,\rho] + \sum_{i,j=1}^{d^2-1} C_{ij} \left(F_i \, \rho \, F_j^\dagger - \frac{1}{2}\left\{ F_j^\dagger F_i , \rho \right\} \right)$$

with $F_{d^2} = \frac{1}{d}$, $\mathrm{Tr}\left(F_j^\dagger F_i\right) = \delta_{ij}$ and $\boxed{C = [C_{ij}] \geq 0}$

$H = H^\dagger$

1.4. Bipartite Entanglement witnessing

For pure states of $S_1 + S_2$ we know that their entanglement is witnessed by the spectrum of the reduced density matrices.
And for density matrices?
How can one check whether a given $\rho \in \mathcal{S}(S_1 + S_2)$ can be convexly expanded as $\rho = \sum_{i,j} \lambda_{ij}\, \rho_{1i} \otimes \rho_{2j}$ or not?

$\boxed{\text{Definition 1.4.1}}$ Partial transposition

Given a bipartite quantum system $S_1 + S_2$ where S_1 and S_2 are described by the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of dimensions d_1 and d_2 and by the matrix algebras $M_{d_1}(\mathbb{C})$ and $M_{d_2}(\mathbb{C})$, let T_1 and T_2 denote the transpositions (with respect to chosen orthonormal bases) on $M_{d_1}(\mathbb{C})$ and $M_{d_2}(\mathbb{C})$ respectively. Then

$T_1 \otimes id_2 : M_{d_1}(\mathbb{C}) \otimes M_{d_2}(\mathbb{C}) \circlearrowleft$ and $id_1 \otimes T_2 : M_{d_1}(\mathbb{C}) \otimes M_{d_2}(\mathbb{C}) \circlearrowleft$

are called Partial Transpositions

Proposition 1.4.1 If $\rho \in \mathcal{S}(S_1+S_2)$ is separable then
$T_1 \otimes id_2[\rho] \geq 0$ and $id_1 \otimes T_2[\rho] \geq 0$.

Proof: $\rho = \sum_{i,j} d_{ij}\, \rho_{1i} \otimes \rho_{2j} \Longrightarrow \begin{cases} T_1 \otimes id_2[\rho] = \sum_{i,j} d_{ij}\, T_1[\rho_{1i}] \otimes \rho_{2j} \geq 0 \\ id_1 \otimes T_2[\rho] = \sum_{i,j} d_{ij}\, \rho_{1i} \otimes T_2[\rho_{2j}] \geq 0 \end{cases}$

Corollary 1.4.1 If $\rho \in \mathcal{S}(S_1+S_2)$ is such that $T_1 \otimes id_2[\rho] \ngeq 0$
then ρ is entangled.

Exercise 1.4.1. Prove that $T_1 \otimes id_2[\rho] \ngeq 0 \Longrightarrow id_1 \otimes T_2[\rho] \ngeq 0$
and viceversa.

Example 1.4.1. $P_{unif} \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$

$$T \otimes id[P_{unif}] = \frac{1}{n} \sum_{i,j=1}^{n} |j\rangle\langle i| \otimes |i\rangle\langle j| \ngeq 0$$

$$id \otimes T[P_{unif}] = \frac{1}{n} \sum_{i,j=1}^{n} |i\rangle\langle j| \otimes |j\rangle\langle i| \ngeq 0$$

Exercise 1.4.2 Prove that the set of separable states in $\mathcal{S}(S_1+S_2)$ is convex.

$$\rho_1 = \sum_{i,j} \lambda^{(1)}_{ij}\, \rho^{(1)}_{1i} \otimes \rho^{(1)}_{2j} \quad , \quad \rho_2 = \sum_{i,j} \lambda^{(2)}_{ij}\, \rho^{(2)}_{1i} \otimes \rho^{(2)}_{2j}$$

$1 \geq \lambda \geq 0$:
$$\lambda \rho_1 + (1-\lambda)\rho_2 = \sum_{i,j} \left(\lambda \lambda^{(1)}_{ij}\, \rho^{(1)}_{1i} \otimes \rho^{(1)}_{2j} + (1-\lambda)\lambda^{(2)}_{ij}\, \rho^{(2)}_{1i} \otimes \rho^{(2)}_{2j} \right)$$

Remark : The subset $\mathcal{S}_{sep}(S_1+S_2) \subset \mathcal{S}(S_1+S_2)$ of separable bipartite states is not only convex, but also closed with respect to the norm topology.

Remark : If all bipartite states in $\mathcal{S}(S_1+S_2)$ were separable then positivity of maps would be sufficient for their physical consistency.

Indeed $\Lambda : \mathcal{S}(S_1+S_2) \to$ if positive :

$$\Lambda \otimes id \left[\sum_{i,j} \lambda_{ij}\, \rho_{1i} \otimes \rho_{2j} \right] = \sum_{ij} \lambda_{ij}\, \Lambda[\rho_{1i}] \otimes \rho_{2j} \geq 0$$

Positive but NOT Completely Positive maps cannot describe physical operations; however they can be used to detect bipartite entanglement.

$\boxed{\text{Exercise 1.4.3}}$ (Generalization of Theorem 1.3.1.)

Show that $\mathcal{L} : M_m(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ $(n \neq m)$ is Completely Positive iff its Choi matrix $M_n(\mathbb{C}) \otimes M_m(\mathbb{C}) \ni M_{\mathcal{L}} := \mathcal{L} \otimes \mathrm{id}_m \left[P^{(m)}_{unif} \right] \geq 0$.

$\boxed{\text{Lemma 1.4.1}}$ $\mathcal{L} : M_m(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ is Positive iff its Choi matrix is block-positive: $\langle \psi \otimes \phi | M_{\mathcal{L}} | \psi \otimes \phi \rangle \geq 0$ $\forall |\psi\rangle \in \mathbb{C}^n, |\phi\rangle \in \mathbb{C}^m$.

Proof: $$\langle \psi \otimes \phi | M_{\mathcal{L}} | \psi \otimes \phi \rangle = \frac{1}{m} \sum_{i,j=1}^{m} \langle \psi | \mathcal{L}[|i\rangle\langle j|] | \psi \rangle \langle j | \phi \rangle \langle \phi | i \rangle$$

$$= \frac{1}{m} \langle \psi | \mathcal{L}[|\phi^*\rangle\langle \phi^*|] | \psi \rangle \quad , \quad \langle i | \phi^* \rangle = \langle \phi | i \rangle = \overline{\langle i | \phi \rangle}$$

i) $\mathcal{L}$ Positive $\Longrightarrow$ $\mathcal{L}[|\phi^*\rangle\langle\phi^*|] \geq 0$ $\forall |\phi\rangle \in \mathbb{C}^m \Longrightarrow M_{\mathcal{L}}$ block-positive

ii) $M_{\mathcal{L}}$ block-positive $\Longrightarrow$ $\mathcal{L}[|\phi^*\rangle\langle\phi^*|] \geq 0$ $\forall |\phi\rangle \in \mathbb{C}^m \Longrightarrow$ $\mathcal{L}$ Positive

$\boxed{\text{Lemma 1.4.2}}$ Let $A \subset \mathbb{R}^d$ be a compact convex subset and $x \in \mathbb{R}^d \setminus A$ a vector <u>not</u> in A. Then, there exist $y_*, x_* \in \mathbb{R}^d$ such that:

(see Hahn-Banach separations theorems)

$$\langle y_* | x \rangle < \langle y_* | x_* \rangle \leq \langle y_* | y \rangle \quad \forall y \in A$$

<u>Proof</u> 1.) $\inf_{y \in A} \|x - y\| = \|x - x_*\| \quad x_* \in A$

2.) set $\boxed{y_* = -x + x_*}$: $\langle y_* | x \rangle = \langle y_* | x_* \rangle - \langle y_* | y_* \rangle < \langle y_* | x_* \rangle$

3.) Let $0 \leq t \leq 1$ and set $y_t = ty + (1-t)x_* \in A$, then

$$0 \leq \|y_t - x\|^2 - \|x_* - x\|^2 = \|t(y - x_*) + x_* - x\|^2 - \|x_* - x\|^2$$

$$= t^2 \|y - x_*\|^2 + 2t\left(\langle y | y_* \rangle - \langle x_* | y_* \rangle\right)$$

$$= o(t) + 2t\left(\langle y | y_* \rangle - \langle x_* | y_* \rangle\right)$$

Letting $t \to 0^+$ one gets $\langle y | y_* \rangle \geq \langle x_* | y_* \rangle \quad \forall y \in A$.

<u>Remark</u>: 1.) Separable states $\rho_{sep} \in M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ can be identified with vectors $y \in \mathbb{R}^{(mn)^2}$ in a closed, convex subset $A \subset \mathbb{R}^d$.

2.) Entangled states $\rho_{ent} \in M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ can be identified with vectors $x \in \mathbb{R}^{(mn)^2} \setminus A$.

3.) Self-adjoint matrices $Z \in M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ can be identified with vectors $z \in \mathbb{R}^{(mn)^2}$ such that

$$\mathrm{Tr}(Z\rho) = \langle\langle Z \mid \rho \rangle\rangle = \langle z \mid r \rangle$$

Lemma 1.4.3 Given an entangled state $\rho_{ent} \in M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ there exists $Y_* \in M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ such that

$$\mathrm{Tr}(Y_* \rho_{ent}) < \alpha \leq \mathrm{Tr}(Y_* \rho_{sep}) \quad \forall \rho_{sep} \in \mathcal{S}(S_1 + S_2).$$

<u>Proof</u>: set $\alpha := \langle x_* \mid x \rangle$ in Lemma 1.4.2 wher $x \in \mathbb{R}^{(mn)^2}$ identifies ρ_{ent}.

$\boxed{\text{Theorem 1.4.1.}}$ (Horodecky)

A bipartite state $\mathcal{S}(S_1+S_2) \ni \rho \in M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ is <u>entangled</u>

iff $\exists$ a Positive, NOT Completely Positive map $\mathcal{L} : M_m(\mathbb{C}) \to M_n(\mathbb{C})$

such that $\mathcal{L} \otimes \mathrm{id}_m[\rho] \ngeq 0$.

<u>Proof:</u> From Lemma 1.4.3: $\mathrm{Tr}\left((Y_* - \alpha)\,\rho_{ent}\right) < 0 \leq \mathrm{Tr}\left((Y_* - \alpha)\,\rho_{sep}\right)$

for all separable ρ_{sep}.

Write $Y_* - \alpha$ as $Y_* - \alpha = \tilde{\mathcal{L}} \otimes \mathrm{id}_m\left[P^{(m)}_{unif}\right]$, then

$\mathrm{Tr}\left(\tilde{\mathcal{L}} \otimes \mathrm{id}_m\left[P^{(m)}_{unif}\right] \rho_{ent}\right) < 0 \implies \tilde{\mathcal{L}}$ <u>NOT Completely Positive</u>

$\mathrm{Tr}\left(\tilde{\mathcal{L}} \otimes \mathrm{id}\left[P^{(m)}_{unif}\right] \rho_{sep}\right) \geq 0 \ \forall \rho_{sep} \implies \tilde{\mathcal{L}} \otimes \mathrm{id}\left[P^{(m)}_{unif}\right]$ <u>block-positive</u>.

Set $\mathcal{L}$ equal to the dual map of $\tilde{\mathcal{L}}$.

Notice $\tilde{\mathcal{L}} : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ and $\mathcal{L} : M_n(\mathbb{C}) \to M_m(\mathbb{C})$

Corollary 1.4.2. (Horodecky) Let $m=2=n$; $m=3, n=2$; $m=2, n=3$,

then, $\rho \in M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ is entangled iff it does not remain positive under partial transposition: $\mathcal{T} \otimes id_m[\rho] \not\geq 0$.

Proof: based on Woronowicz theorem which states that

any positive map $\tilde{\mathcal{L}} : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ $(m=n=2;\ m=2, n=3;\ m=3, n=2)$ can be written as $\boxed{\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_1 + \mathcal{T} \circ \tilde{\mathcal{L}}_2}$ where $\boxed{\mathcal{L}_{1,2} : M_m(\mathbb{C}) \to M_n(\mathbb{C})}$ are completely positive maps and $\mathcal{T} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is the transposition.

- $\mathcal{T} \otimes id_m[\rho] \not\geq 0 \implies \rho$ entangled (see Corollary 1.4.1)
- ρ entangled $\implies \exists\ \tilde{\mathcal{L}}$ positive from $M_m(\mathbb{C}) \to M_n(\mathbb{C})$ whose dual $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 \circ \mathcal{T} : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ is such that $\mathcal{L} \otimes id_m[\rho] = \mathcal{L}_1 \otimes id_m[\rho] + (\mathcal{L}_1 \otimes id_m) \circ (\mathcal{T} \otimes id_m)[\rho] \not\geq 0$

Then, $\mathcal{T} \otimes id_m[\rho] \not\geq 0$ since $\mathcal{L}_1 \otimes id_m$ is positive.

Remark: in low dimension $(m = n = 2\,;\ m = 2, n = 3\,;\ m = 3, n = 2)$ separable states are PPT, namely Positive under Partial Transposition.

Remark: Because of Woronowicz theorem, in low dimension Positive maps have a definite decomposable structure: $\mathcal{L} = \mathcal{L}_1 + T \circ \mathcal{L}_2$. However, already with $m = n = 3$ there are Positive map $\mathcal{L}$ which are <u>NOT</u> decomposable.

$\boxed{\text{Exercise 15.4}}$ Show that any positive map $\mathcal{L} : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ with $m = n = 2\,;\ m = 3, n = 2\,;\ m = 2, n = 3$, can also be written as $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 \circ T$ where $\mathcal{L}_{1,2} : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ are completely positive and $T : M_m(\mathbb{C}) \to M_m(\mathbb{C})$ is the transposition. (Use that $T \circ T = id$)