LESSON 14.

1. Product of quasi–projective varieties and tensors.

1.1. **Products.** Let $\mathbb{P}^n$, $\mathbb{P}^m$ be projective spaces over the same field K. The cartesian product $\mathbb{P}^n \times \mathbb{P}^m$ is simply a set: we want to define an injective map from $\mathbb{P}^n \times \mathbb{P}^m$ to a suitable projective space, so that the image is a projective variety, which will be identified with our product.

Let $N = (n+1)(m+1) - 1$ and define $\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ in the following way: $\sigma([x_0, \dots, x_n], [y_0, \dots, y_m]) = [x_0y_0, x_0y_1, \dots, x_iy_j, \dots, x_ny_m]$. Using coordinates w_{ij}, $i = 0, \dots, n$, $j = 0, \dots, m$, in $\mathbb{P}^N$, σ is defined by

$$\{w_{ij} = x_iy_j,\ i = 0, \dots, n, j = 0, \dots, m.$$

It is easy to observe that σ is a well–defined map.

Let $\Sigma_{n,m}$ (or simply Σ) denote the image $\sigma(\mathbb{P}^n \times \mathbb{P}^m)$.

Proposition 1.1. *σ is injective and $\Sigma_{n,m}$ is a closed subset of $\mathbb{P}^N$.*

Proof. If $\sigma([x],[y]) = \sigma([x'],[y'])$, then there exists $\lambda \neq 0$ such that $x'_iy'_j = \lambda x_iy_j$ for all i, j. In particular, if $x_h \neq 0$, $y_k \neq 0$, then also $x'_h \neq 0$, $y'_k \neq 0$, and for all i $x'_i = \lambda \frac{y_k}{y'_k} x_i$, so $[x_0, \dots, x_n] = [x'_0, \dots, x'_n]$. Similarly for the second point.
To prove the second assertion, I claim: $\Sigma_{n,m}$ is the closed set of equations:

$$\{w_{ij}w_{hk} = w_{ik}w_{hj},\ i, h = 0, \dots, n; j, k = 0 \dots, m. \tag{1}$$

It is clear that if $[w_{ij}] \in \Sigma$, then it satisfies (1).

Conversely, assume that $[w_{ij}]$ satisfies (1) and that $w_{\alpha\beta} \neq 0$. Then

$$\begin{aligned}[w_{00}, \dots, w_{ij}, \dots, w_{nm}] &= [w_{00}w_{\alpha\beta}, \dots, w_{ij}w_{\alpha\beta}, \dots, w_{nm}w_{\alpha\beta}] = \\ &= [w_{0\beta}w_{\alpha 0}, \dots, w_{i\beta}w_{\alpha j}, \dots, w_{n\beta}w_{\alpha m}] = \\ &= \sigma([w_{0\beta}, \dots, w_{n\beta}], [w_{\alpha 0}, \dots, w_{\alpha m}]).\end{aligned}$$

□

σ is called the Segre map and $\Sigma_{n,m}$ the Segre variety or biprojective space. Note that Σ is covered by the affine open subsets $\Sigma^{ij} = \Sigma \cap W_{ij}$, where $W_{ij} = \mathbb{P}^N \setminus V_P(w_{ij})$. Moreover $\Sigma^{ij} = \sigma(U_i \times V_j)$, where $U_i \times V_j$ is naturally identified with $\mathbb{A}^{n+m}$.

Proposition 1.2. *$\sigma|_{U_i \times V_j} : U_i \times V_j = \mathbb{A}^{n+m} \to \Sigma^{ij}$ is an isomorphism of varieties.*

Proof. Assume by simplicity $i = j = 0$. Choose non–homogeneous coordinates on U_0: $u_i = x_i/x_0$ and on V_0: $v_j = y_j/y_0$. So $u_1, \dots u_n, v_1, \dots, v_m$ are coordinates on $U_0 \times V_0$. Take non–homogeneous coordinates also on W_{00}: $z_{ij} = w_{ij}/w_{00}$.

Using these coordinates we have:

$$\begin{array}{rl} \sigma|_{U_i \times V_j} : & (u_1, \dots u_n, v_1, \dots, v_m) \to (v_1, \dots, v_m, u_1, u_1v_1, \dots, u_1v_m, \dots, u_nv_m) \\ & \quad || \\ & ([1, u_1, \dots, u_n], [1, v_1, \dots, v_m]) \end{array}$$

i.e. $\sigma(u_1, \dots, v_m) = (z_{01}, \dots, z_{nm})$, where

$$\begin{cases} z_{i0} = u_i, & \text{if } i = 1, \dots, n; \\ z_{0j} = v_j, & \text{if } j = 1, \dots, m; \\ z_{ij} = u_iv_j = z_{i0}z_{0j} & \text{otherwise.} \end{cases}$$

Hence $\sigma|_{U_0 \times V_0}$ is regular.
The inverse map takes $(z_{01}, \dots, z_{nm})$ to $(z_{10}, \dots, z_{n0}, z_{01}, \dots, z_{0m})$, so it is also regular. □

Corollary 1.3. *$\mathbb{P}^n \times \mathbb{P}^m$ is irreducible and birational to $\mathbb{P}^{n+m}$.*

Proof. The first assertion follows from Ex.5, Lesson 7, considering the covering of Σ by the open subsets Σ^{ij}. Indeed, $\Sigma^{ij} \cap \Sigma^{hk} = \sigma((U_i \times V_j) \cap (U_h \times V_k)) = \sigma((U_i \cap U_h) \times (V_j \cap V_k))$, and $U_i \cap U_h \neq \emptyset \neq V_j \cap V_k$.

For the second assertion, by Theorem 1.6, Lesson 13, it is enough to note that $\Sigma_{n,m}$ and $\mathbb{P}^{n+m}$ contain isomorphic open subsets, i.e. Σ^{ij} and $\mathbb{A}^{n+m}$. □

From now on, we shall identify $\mathbb{P}^n \times \mathbb{P}^m$ with $\Sigma_{n,m}$. If $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$ are any quasi–projective varieties, then $X \times Y$ will be automatically identified with $\sigma(X \times Y) \subset \Sigma$.

Proposition 1.4. *If X and Y are projective varieties (resp. quasi–projective varieties), then $X \times Y$ is projective (resp. quasi–projective).*

Proof.

$$\begin{aligned} \sigma(X \times Y) &= \bigcup_{i,j} (\sigma(X \times Y) \cap \Sigma^{ij}) = \\ &= \bigcup_{i,j} (\sigma(X \times Y) \cap (U_i \times V_j)) = \\ &= \bigcup_{i,j} (\sigma((X \cap U_i) \times (Y \cap V_j))). \end{aligned}$$

If X and Y are projective varieties, then $X \cap U_i$ is closed in U_i and $Y \cap V_j$ is closed in V_j, so their product is closed in $U_i \times V_j$; since $\sigma|_{U_i \times V_j}$ is an isomorphism, also $\sigma(X \times Y) \cap \Sigma^{ij}$ is closed in Σ^{ij}, so $\sigma(X \times Y)$ is closed in Σ, by Lemma 1.3, Lesson 10.

If X, Y are quasi–projective, the proof is similar: $X \cap U_i$ is locally closed in U_i and $Y \cap V_j$ is locally closed in V_j, so $X \cap U_i = Z \setminus Z'$, $Y \cap V_j = W \setminus W'$, with Z, Z', W, W' closed. Therefore $(Z \setminus Z') \times (W \setminus W') = Z \times W \setminus ((Z' \times W) \cup (Z \times W'))$, which is locally closed.

As for the irreducibility, see Exercise 1, this Lesson. □

Example 1.5. $\mathbb{P}^1 \times \mathbb{P}^1$

The example of $\mathbb{P}^1 \times \mathbb{P}^1$, the Segre quadric, has already been studied in Lesson 3, 1.5.

We recall that $\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ is given by the parametric equations $\{w_{ij} = x_i y_j,\ i = 0, 1,$ $j = 0, 1$. Σ has only one non–trivial equation: $w_{00}w_{11} - w_{01}w_{10}$, hence Σ is a quadric. The equation of Σ can be written as

$$\begin{vmatrix} w_{00} & w_{01} \\ w_{10} & w_{11} \end{vmatrix} = 0. \tag{2}$$

Σ contains two families of special closed subsets parametrised by $\mathbb{P}^1$, i.e.

$$\{\sigma(\{P\} \times \mathbb{P}^1)\}_{P \in \mathbb{P}^1} \quad \text{and} \quad \{\sigma(\mathbb{P}^1 \times \{Q\})\}_{Q \in \mathbb{P}^1}.$$

If $P = [a_0, a_1]$, then $\sigma(\{P\} \times \mathbb{P}^1)$ is given by the equations:

$$\begin{cases} w_{00} = a_0 y_0 \\ w_{01} = a_0 y_1 \\ w_{10} = a_1 y_0 \\ w_{11} = a_1 y_1 \end{cases}$$

hence it is a line. Cartesian equations of $\sigma(\{P\} \times \mathbb{P}^1)$ are:

$$\begin{cases} a_1 w_{00} - a_0 w_{10} = 0 \\ a_1 w_{01} - a_0 w_{11} = 0; \end{cases}$$

they express the proportionality of the rows of the matrix (2) with coefficients $[a_1, -a_0]$. Similarly, $\sigma(\mathbb{P}^1 \times \{Q\})$ is the line of equations

$$\begin{cases} a_1 w_{00} - a_0 w_{01} = 0 \\ a_1 w_{10} - a_0 w_{11} = 0. \end{cases}$$

Hence Σ contains two families of lines, called the rulings of Σ: two lines of the same ruling are clearly disjoint while two lines of different rulings intersect at one point $(\sigma(P, Q))$. Conversely, through any point of Σ there pass two lines, one for each ruling. Note that Σ is exactly the quadric surface of Lesson 13, Example 1.9.d) and that the projection of centre $[1,0,0,0]$ realizes an explicit birational map between $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$.

1.2. **Tensors.** The product of projective spaces has a coordinate-free description in terms of tensors. Precisely, let $\mathbb{P}^n = \mathbb{P}(V)$ and $\mathbb{P}^m = \mathbb{P}(W)$. The tensor product $V \otimes W$ of the vector spaces V, W is constructed as follows: let $K(V \times W)$ be the K-vector space with basis $V \times W$ obtained as the set of formal finite linear combinations of type $\Sigma_i a_i(v_i, w_i)$ with $a_i \in K$. Let U be the vector subspace generated by all elements of the form:

$(v, w) + (v', w) - (v + v', w)$,
$(v, w) + (v, w') - (v, w + w')$,
$(\lambda v, w) - \lambda(v, w)$,
$(v, \lambda w) - (\lambda(v, w)$,

with $v, v' \in V$, $w, w' \in W$, $\lambda \in K$. The tensor product is by definition the quotient $V \otimes W := K(V \times W)/U$. The class of a pair (v, w) is denoted by $v \otimes w$, and called a decomposable tensor. So $V \otimes W$ is generated by the decomposable tensors; more precisely, a general element $\omega \in V \otimes W$ is of the form $\Sigma_{i=1}^k v_i \otimes w_i$, with $v_i \in V$, $w_i \in W$. The minimum k such that an expression of this type exists is called the tensor rank of ω.

There is a natural bilinear map $\otimes : V \times W \to V \otimes W$, such that $(v, w) \to v \otimes w$. It enjoys the following universal property: for any K-vector space Z with a bilinear map $f : V \times W \to Z$, there exists a unique linear map $\bar{f} : V \otimes W \to Z$ such that f factorizes in the form $f = \bar{f} \circ \otimes$.

If $\dim V = n$, $\dim W = m$, and bases $\mathcal{B} = (e_1, \dots, e_n), \mathcal{B}' = (e'_1, \dots, e'_m)$ are given, then $(e_1 \otimes e'_1, \dots, e_i \otimes e'_j, \dots e_n \otimes e'_m)$ is a basis of $V \otimes W$: therefore $\dim V \otimes W = nm$.

If $v = x_1 e_1 + \dots x_n e_n$, $w = y_1 e'_1 + \dots y_m e'_m$, then $v \otimes w = \Sigma x_i y_j e_i \otimes e'_j$. So, passing to the projective spaces, the map $\otimes$ defines precisely the Segre map

$$\sigma : \mathbb{P}(V) \times \mathbb{P}(W) \to \mathbb{P}(V \otimes W), \quad ([v], [w]) \to [v \otimes w].$$

Indeed in coordinates we have $([x_0, \dots, x_n], [y_0, \dots, y_m]) \to [w_{00}, \dots, w_{nm}]$, with $w_{ij} = x_i y_j$. The image of $\otimes$ is the set of decomposable tensors, or rank one tensors.

The tensor product $V \otimes W$ has the same dimension, and is therefore isomorphic to the vector space of $n \times m$ matrices. The coordinates w_{ij} can be interpreted as the entries of such a $n \times m$ matrix. The equations of the Segre variety $\Sigma_{n,m}$ are the 2×2 minors of the matrix, therefore $\Sigma_{n,m}$ can be interpreted as the set of matrices of rank one.

The construction of the tensor product can be iterated, to construct $V_1 \otimes V_2 \otimes \dots \otimes V_r$. The following properties can easily be proved:

1. $V_1 \otimes (V_2 \otimes V_3) \simeq (V_1 \otimes V_2) \otimes V_3$;
2. $V \otimes W \simeq W \otimes V$;
3. $V^* \otimes W \simeq Hom(V, W)$, where $f \otimes w \to (V \to W : v \to f(v)w)$.

Also the Veronese morphism has a coordinate free description, in terms of symmetric tensors. Given a vector space V, for any $d \geq 0$ the d-th symmetric power of V, S^dV or Sym^dV, is constructed as follows. We consider the tensor product of d copies of V: $V \otimes \cdots \otimes V = V^{\otimes d}$, and we consider its subvector space U generated by tensors of the form $v_1 \otimes \ldots v_d - v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(d)}$, where σ varies in the symmetric group on d elements $\mathcal{S}_d$. Then by definition $S^dV := V^{\otimes d}/U$. The equivalence class $[v_1 \otimes \cdots \otimes v_d]$ is denoted as a product $v_1 \ldots v_d$.

There is a natural multilinear and symmetric map $V \times \cdots \times V = V^d \to S^dV$, such that $(v_1, \ldots, v_d) \to v_1 \ldots v_d$, which enjoys the universal property. S^dV is generated by the products $v_1 \ldots v_d$.

In characteristic 0, S^dV can also be interpreted as a subspace of $V^{\otimes d}$, by considering the following map, that is an isomorphism to the image:

$$S^dV \to V^{\otimes d}, \quad v_1 \ldots v_d \to \Sigma_{\sigma \in \mathcal{S}_d} \frac{1}{d!} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}.$$

If $\mathcal{B} = (e_1, \ldots, e_n)$ is a basis of V, then it is easy to check that a basis of S^dV is formed by the monomials of degree d in $e_1, \ldots, e_n$; therefore $\dim S^dV = \binom{n+d-1}{d}$.

For instance, in S^2V the product v_1v_2 can be identified with $\frac{1}{2}(v_1 \otimes v_2 + v_2 \otimes v_1)$.

The symmetric algebra of V is $SV := \oplus_{d \geq 0} S^dV = K \oplus V \oplus S^2V \oplus \ldots$. An inner product can be naturally defined to give it the structure of a K-algebra, which results to be isomorphic to the polynomial ring in n variables, where $n = \dim V$.

If $charK = 0$ the Veronese morphism can be interpreted in the following way:

$$v_{n,d} : \mathbb{P}(V) \to \mathbb{P}(S^dV), \ [v] = [x_0e_0 + \ldots x_ne_n] \to [v^d] = [(x_0e_0 + \cdots + x_ne_n)^d].$$

Moreover S^2V can be interpreted as the space of symmetric $d \times d$ matrices, and the Veronese variety $V_{n,2}$ as the subset of the symmetric matrices of rank one.

Exercises 1.6. 1. Using Ex. 5 of Lesson 7, prove that, if $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$ are irreducible projective varieties, then $X \times Y$ is irreducible.

2. Let L, M, N be the following lines in $\mathbb{P}^3$:

$$L : x_0 = x_1 = 0, M : x_2 = x_3 = 0, N : x_0 - x_2 = x_1 - x_3 = 0.$$

Let X be the union of lines meeting L, M and N: write equations for X and describe it: is it a projective variety? If yes, of what dimension and degree?

3. Let X, Y be quasi–projective varieties, identify $X \times Y$ with its image via the Segre map. Check that the two projection maps $X \times Y \xrightarrow{p_1} X$, $X \times Y \xrightarrow{p_2} Y$ are regular. (Hint: use the open covering of the Segre variety by the Σ^{ij}'s.)