LESSON 13.

1. Rational maps

Let X, Y be quasi–projective varieties over an algebraically closed field K. The idea to define rational maps is that they are to regular maps as rational functions are to regular functions.

Definition 1.1. The *rational maps* from X to Y are the germs of regular maps from open subsets of X to Y, i.e. they are equivalence classes of pairs (U, φ), where $U \neq \emptyset$ is open in X and $\varphi : U \to Y$ is regular. The equivalence relation is of course defined by $(U, \varphi) \sim (V, \psi)$ if and only if $\varphi|_{U\cap V} = \psi|_{U\cap V}$.

We need to prove that this is indeed an equivalence relation. The following Lemma guarantees that this is the case.

Lemma 1.2. *Let $\varphi, \psi : X \to Y \subset \mathbb{P}^n$ be regular maps between quasi-projective varieties. If $\varphi|_U = \psi|_U$ for $U \subset X$ open and non–empty, then $\varphi = \psi$.*

Proof. Let $P \in X$ and consider $\varphi(P)$, $\psi(P) \in Y$. There exists a hyperplane H such that $\varphi(P) \notin H$ and $\psi(P) \notin H$ (otherwise the dual projective space $\check{\mathbb{P}}^n$ would be the union of its two hyperplanes $H_{\varphi(P)}$, $H_{\psi(P)}$, defined by the conditions of containing respectively $\varphi(P)$ and $\psi(P)$).

Up to a projective transformation, we can assume that $H = V_P(x_0)$, so $\varphi(P), \psi(P) \in U_0$. Set $V = \varphi^{-1}(U_0) \cap \psi^{-1}(U_0)$: an open neighbourhood of P. Consider the restrictions of φ and ψ from V to $Y \cap U_0$: they are regular maps whose codomain is contained in $U_0 \simeq \mathbb{A}^n$. Since they coincide on $V \cap U$, their components φ_i, ψ_i, $i = 1, \dots, n$, coincide on $V \cap U$, hence on V (Corollary 1.4, 2, Lesson 10). So $\varphi_i|_V = \psi_i|_V$. In particular $\varphi(P) = \psi(P)$. $\square$

A rational map from X to Y will be denoted by $\varphi : X \dashrightarrow Y$. As for rational functions, the domain of definition of φ, dom φ, is the maximum open subset of X such that φ is regular at the points of dom φ.

The following proposition follows from the characterization of rational functions on affine varieties.

Proposition 1.3. *Let X, Y be affine algebraic sets, with Y closed in $\mathbb{A}^n$. Then $\varphi : X \dashrightarrow Y$ is a rational map if and only if $\varphi = (\varphi_1, \dots, \varphi_n)$, where $\varphi_1, \dots, \varphi_n \in K(X)$.*

If $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$, then a rational map $X \dashrightarrow Y$ is assigned by giving $m+1$ homogeneous polynomials of $K[x_0, x_1, \dots, x_n]$ of the same degree, $F_0, \dots, F_m$, such that *at least one* of them is not identically zero on X.

A rational map $\varphi : X \dashrightarrow Y$ is called *dominant* if the image of X via φ is dense in Y, i.e. if $\overline{\varphi(U)} = Y$, where $U = \text{dom } \varphi$.

Dominant rational maps can be composed: if $\varphi : X \dashrightarrow Y$ is dominant and $\psi : Y \dashrightarrow Z$ is any rational map, then dom $\psi \cap \text{Im}\varphi \neq \emptyset$, so we can define $\psi \circ \varphi : X \dashrightarrow Z$: it is the germ of the map $\psi \circ \varphi$, regular on $\varphi^{-1}(\text{dom } \psi \cap \text{Im}\varphi)$. We note that also the composed rational map $\psi \circ \varphi$ is dominant.

Definition 1.4. A *birational map* from X to Y is a rational map $\varphi : X \dashrightarrow Y$ such that φ is dominant and there exists $\psi : Y \dashrightarrow X$, a dominant rational map, such that $\psi \circ \varphi = 1_X$ and $\varphi \circ \psi = 1_Y$ **as rational maps**. In this case, X and Y are called *birationally equivalent* or simply *birational*.

If $\varphi : X \dashrightarrow Y$ is a dominant rational map, then we can define the comorphism $\varphi^* : K(Y) \to K(X)$ in the usual way: it is an injective K–homomorphism.

Proposition 1.5. *Let X, Y be quasi–projective varieties, and let $u : K(Y) \to K(X)$ be a K–homomorphism. Then there exists a rational map $\varphi : X \dashrightarrow Y$ such that $\varphi^* = u$.*

Proof. Y is covered by open affine varieties Y_α, $\alpha \in I$ (by Proposition 1.12, Lesson 11); note that for any index α, $K(Y) \simeq K(Y_\alpha)$ (Prop. 1.9, Lesson 10) and $K(Y_\alpha) \simeq K(t_1, \dots, t_n)$, where $t_1, \dots, t_n$ can be interpreted as coordinate functions on Y_α. Choose such an open subset Y_α. Then $u(t_1), \dots, u(t_n) \in K(X)$ and there exists $U \subset X$, non–empty open subset such that $u(t_1), \dots, u(t_n)$ are all regular on U. So $u(K[t_1, \dots, t_n]) \subset \mathcal{O}(U)$ and we can consider the regular map $u^\sharp : U \to Y_\alpha \hookrightarrow Y$. The germ of $u^\sharp$ gives a rational map $X \dashrightarrow Y$. It is possible to check that this rational map does not depend on the choice of Y_α and U. □

Theorem 1.6. *Let X, Y be quasi–projective varieties. The following are equivalent:*
(i) X is birational to Y;
(ii) $K(X) \simeq K(Y)$;
(iii) there exist non–empty open subsets $U \subset X$ and $V \subset Y$ such that $U \simeq V$.

Proof. (i) $\Leftrightarrow$ (ii) via the construction of the comorphism φ^* associated to φ and of $u^\sharp$, associated to $u : K(Y) \to K(X)$. One checks that both constructions are functorial.

(i) $\Rightarrow$ (iii) Let $\varphi : X \dashrightarrow Y$, $\psi : Y \dashrightarrow X$ be rational maps inverse each other. Put $U' = \text{dom } \varphi$ and $V' = \text{dom } \psi$. By assumption, $\psi \circ \varphi$ is defined on $\varphi^{-1}(V')$ and coincides with 1_X there. Similarly, $\varphi \circ \psi$ is defined on $\psi^{-1}(U')$ and equal to 1_Y. Then φ and ψ establish an isomorphism between the corresponding sets $U := \varphi^{-1}(\psi^{-1}(U'))$ and $V := \psi^{-1}(\varphi^{-1}(V'))$.

(iii) $\Rightarrow$ (ii) $U \simeq V$ implies $K(U) \simeq K(V)$; but $K(U) \simeq K(X)$ and $K(V) \simeq K(Y)$ (Prop. 1.9, Lesson 10), so $K(X) \simeq K(Y)$ by transitivity. $\square$

Corollary 1.7. *If X is birational to Y, then* $\dim X = \dim Y$.

Corollary 1.8. *The projective space $\mathbb{P}^n$ and the affine space $\mathbb{A}^n$ are birationally equivalent.*

Theorem 1.6 can be given an interpretation in the language of categories. We can define a category $\mathcal{C}$ whose objects are the irreducible algebraic varieties over a fixed algebraically closed field K, and the morphisms are the dominant rational maps. The isomorphisms in $\mathcal{C}$ are birational maps, so two objects are isomorphic in $\mathcal{C}$ if they are birationally equivalent. We can consider also the category $\mathcal{C}'$ with objects the fields, finitely generated extensions of K, and morphisms the K-homomorphisms. Then there is a contravariant functor $\mathcal{C} \to \mathcal{C}'$ associating to a variety X its field of rational functions $K(X)$ and to a rational map $\varphi : X \dashrightarrow Y$ its comorphism φ^*. Proposition 1.5 and Theorem 1.6 say that this functor is an equivalence of categories.

There are two classification problems for algebraic varieties, up to isomorphism and up to birational equivalence. Both are central problems of Algebraic Geometry.

Example 1.9.

a) The cuspidal cubic $Y = V(x^3 - y^2) \subset \mathbb{A}^2$.

We have seen that Y is not isomorphic to $\mathbb{A}^1$, but in fact Y and $\mathbb{A}^1$ are birational. Indeed, the regular map $\varphi : \mathbb{A}^1 \to Y$, $t \to (t^2, t^3)$, admits a rational inverse $\psi : Y \dashrightarrow \mathbb{A}^1$, $(x, y) \to \frac{y}{x}$. ψ is regular on $Y \setminus \{(0,0)\}$, ψ is dominant and $\psi \circ \varphi = 1_{\mathbb{A}^1}$, $\varphi \circ \psi = 1_Y$ as rational maps. In particular, $\varphi^* : K(Y) \to K(X)$ is a field isomorphism. Recall that $K[Y] = K[t_1, t_2]$, with $t_1^2 = t_2^3$, so $K(Y) = K(t_1, t_2) = K(t_2/t_1)$, because $t_1 = (t_2/t_1)^2 = t_2^2/t_1^2 = t_1^3/t_1^2$ and $t_2 = (t_2/t_1)^3 = t_2^3/t_1^3 = t_2^3/t_2^2$, so $K(Y)$ is generated by a unique transcendental element. Notice that φ and ψ establish isomorphisms between $\mathbb{A}^1 \setminus \{0\}$ and $Y \setminus \{(0,0)\}$.

b) *Rational maps from $\mathbb{P}^1$ to $\mathbb{P}^n$.*

Let $\varphi : \mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ be a rational map: on some open $U \subset \mathbb{P}^1$,

$$\varphi([x_0, x_1]) = [F_0(x_0, x_1), \dots, F_n(x_0, x_1)],$$

with $F_0, \dots, F_n$ homogeneous of the same degree, without non–trivial common factors. Assume that $F_i(P) = 0$ for a certain index i, with $P = [a_0, a_1]$. Then $F_i \in I_h(P) = \langle a_1x_0 - a_0x_1 \rangle$, i.e. $a_1x_0 - a_0x_1$ is a factor of F_i. This remark implies that $\forall\, Q \in \mathbb{P}^1$ there exists $i \in \{0, \dots, n\}$ such that $F_i(Q) \neq 0$, because otherwise $F_0, \dots, F_n$ would have a common factor of degree 1. Hence we conclude that φ is regular.

We have obtained that any rational map from $\mathbb{P}^1$ is in fact regular.

c) *Projections.*

Let $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ be a rational map, that can be represented in matrix form by $Y = AX$, where A is a $(m+1)\times(n+1)$-matrix, with entries in K. Then φ is a rational map, regular on $\mathbb{P}^n \setminus \mathbb{P}(\mathrm{Ker}A)$. Put $\Lambda := \mathbb{P}(\mathrm{Ker}A)$. If $A = (a_{ij})$, this means that Λ has cartesian equations

$$\begin{cases} a_{00}x_0 + \ldots + a_{0n}x_n = 0 \\ a_{10}x_0 + \ldots + a_{1n}x_n = 0 \\ \ldots \\ a_{m0}x_0 + \ldots + a_{mn}x_n = 0. \end{cases}$$

The map φ has a geometric interpretation: it can be seen as the *projection of centre* Λ to a complementar linear space. To see how to give this interpretation, first of all we can assume that rk $A = m+1$, otherwise we replace $\mathbb{P}^m$ with $\mathbb{P}(\mathrm{Im}\ A)$; hence $\dim\Lambda = (n+1)-(m+1)-1 = n-m-1$.

Consider first the special case in which $\Lambda : x_0 = \cdots = x_m = 0$; we can identify $\mathbb{P}^m$ with the subspace of $\mathbb{P}^n$ of equations $x_{m+1} = \cdots = x_n = 0$, so Λ and $\mathbb{P}^m$ are complementar subspaces, i.e. $\Lambda \cap \mathbb{P}^m = \emptyset$ and the linear span of Λ and $\mathbb{P}^m$ is $\mathbb{P}^n$. Then, for $Q[a_0, \ldots, a_n] \in \mathbb{P}^n \setminus \Lambda$, $\varphi(Q) = [a_0, \ldots, a_m, 0, \ldots, 0]$: it is the intersection of $\mathbb{P}^m$ with the linear span $\overline{\Lambda Q}$ of Λ and Q. In fact, $\overline{\Lambda Q}$ has equations

$$\{a_i x_j - a_j x_i = 0, i, j = 0, \ldots, m \quad \text{(check!)}$$

so $\overline{\Lambda Q} \cap \mathbb{P}^m$ has coordinates $[a_0, \ldots, a_m, 0, \ldots, 0]$.

In the general case, if $\Lambda = V_P(L_0, \ldots, L_m)$, with $L_0, \ldots, L_m$ linearly independent forms, we can identify $\mathbb{P}^m$ with $V_P(L_{m+1}, \ldots, L_n)$, where $L_{m+1}, \ldots, L_n$ are linearly independents linear forms chosen so that $L_0, \ldots, L_m$, $L_{m+1}, \ldots, L_n$ is a basis of $(K^{n+1})^*$. Then $L_0, \ldots, L_m$ can be interpreted as coordinate functions on $\mathbb{P}^m$.

If $m = n-1$, then Λ is a point P and φ, often denoted by π_P, is the projection from P to a hyperplane not containing P. Also for the projection with centre Λ often the notation π_Λ is used.

d)*Rational and unirational varieties.*

A quasi–projective variety X is called *rational* if it is birational to a projective space $\mathbb{P}^n$, or equivalently to $\mathbb{A}^n$.

By Theorem 1.6, X is rational if and only if $K(X) \simeq K(\mathbb{P}^n) = K(x_1, \ldots, x_n)$ for some n, i.e. $K(X)$ is an extension of K generated by a transcendence basis; this kind of extension is called a *purely transcendental extension of* K. In an equivalent way, X is rational if there exists a rational map $\varphi : \mathbb{P}^n \dashrightarrow X$ which is dominant and is an isomorphism if restricted to a suitable open subset $U \subset \mathbb{P}^n$. Hence X admits a *birational parameterization* by polynomials in n parameters.

A weaker notion is that of *unirational* variety: X is unirational if there exists a dominant rational map $\mathbb{P}^n \dashrightarrow X$ i.e. if $K(X)$ is contained in the quotient field of a polynomial ring. Hence X can be parameterized by polynomials, but not necessarily generically one–to–one.

It is clear that, if X is rational, then it is unirational. The converse implication has been an important open problem, up to 1971, when it has been solved in the negative, for varieties of dimension ≥ 3 (Clemens–Griffiths, Iskovskih–Manin, Artin-Mumford). Nevertheless rationality and unirationality are equivalent for curves (Theorem of Lüroth, 1880, over any field) and for surfaces if $\mathrm{char} K = 0$ (Theorem of Castelnuovo, 1894).

As an example of rational variety with an explicit rational parameterization constructed geometrically, let us consider the Segre quadric in $\mathbb{P}^3$, of maximal rank: $X = V_P(x_0x_3 - x_1x_2)$, an irreducible hypersurface of degree 2. Let $\pi_P : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ be the projection of centre $P[1,0,0,0]$, such that $\pi_P([y_0, y_1, y_2, y_3]) = [y_1, y_2, y_3]$. The restriction of π_P to X is a rational map $\tilde{\pi}_P : X \dashrightarrow \mathbb{P}^2$, regular on $X \setminus \{P\}$. $\tilde{\pi}_P$ has a rational inverse: indeed consider the rational map $\psi : \mathbb{P}^2 \dashrightarrow X$, $[y_1, y_2, y_3] \to [y_1y_2, y_1y_3, y_2y_3, y_3^2]$. The equation of X is satisfied by the points of $\psi(\mathbb{P}^2)$: $(y_1y_2)y_3^2 = (y_1y_3)(y_2y_3)$. ψ is regular on $\mathbb{P}^2 \setminus V_P(y_1y_2, y_3)$. Let us compose ψ and $\tilde{\pi}_P$:

$$[y_0, \dots, y_3] \in X \xrightarrow{\pi_P} [y_1, y_2, y_3] \xrightarrow{\psi} [y_1y_2, y_1y_3, y_2y_3, y_3^2];$$

$y_1y_2 = y_0y_3$ implies $\psi \circ \pi_P = 1_X$. In the opposite order:

$$[y_1, y_2, y_3] \xrightarrow{\psi} [y_1y_2, y_1y_3, y_2y_3, y_3^2] \xrightarrow{\pi_P} [y_1y_3, y_2y_3, y_3^2] = [y_1, y_2, y_3].$$

So X is birational to $\mathbb{P}^2$ hence it is a rational surface.

Note that if we consider another projection $\pi_{P'}$ whose centre P' is not on the quadric, we get a regular $2:1$ map to the plane, that is certainly not birational.

e) *A birational non–regular map from $\mathbb{P}^2$ to $\mathbb{P}^2$.*

The following rational map is called the *standard quadratic transformation*:

$$Q : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad [x_0, x_1, x_2] \to [x_1x_2, x_0x_2, x_0x_1].$$

Q is regular on $U := \mathbb{P}^2 \setminus \{A, B, C\}$, where $A[1,0,0]$, $B[0,1,0]$, $C[0,0,1]$ are the fundamental points (see Figure 1).

Let a be the line through B and C: $a = V_P(x_0)$, and similarly $b = V_P(x_1)$, $c = V_P(x_2)$. Then $Q(a) = A$, $Q(b) = B$, $Q(c) = C$. Outside these three lines Q is an isomorphism.

Precisely, put $U' = \mathbb{P}^2 \setminus \{a \cup b \cup c\}$; then $Q : U' \to \mathbb{P}^2$ is regular, the image is U' and $Q^{-1} : U' \to U'$ coincides with Q. Indeed,

$$[x_0, x_1, x_2] \xrightarrow{Q} [x_1x_2, x_0x_2, x_0x_1] \xrightarrow{Q} [x_0^2x_1x_2, x_0, x_1^2x_2, x_0x_1x_2^2].$$

So $Q \circ Q = 1_{\mathbb{P}^2}$ as rational map, hence Q is birational and $Q = Q^{-1}$.

Note that another way to express Q is the following: $[x_0, x_1, x_2] \to [\frac{1}{x_0}, \frac{1}{x_1}, \frac{1}{x_2}]$.

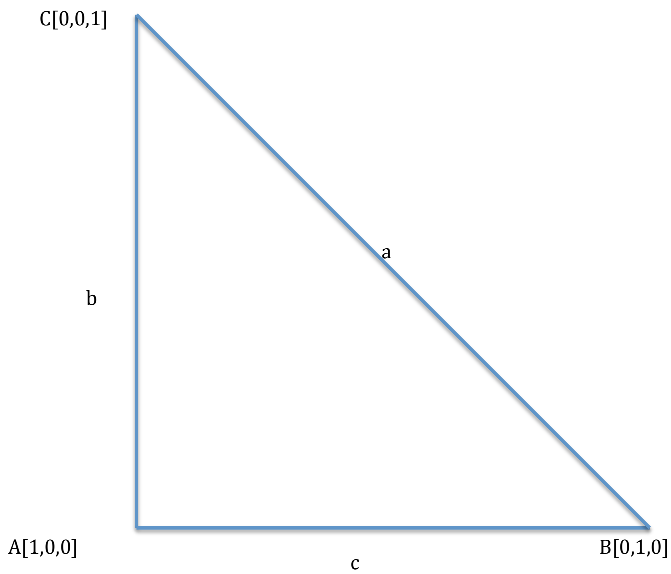

FIGURE 1

The set of the birational maps $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a group, called the *Cremona group*. At the end of XIX century, Max Noether proved that the Cremona group is generated by $PGL(3, K)$ and by the single standard quadratic transformation Q above. The analogous groups for $\mathbb{P}^n$, $n \geq 3$, are much more complicated and a complete description is still unknown.

We conclude this Lesson with a theorem illustrating an application of the linearisation procedure. We shall use the following notation: given a homogeneous polynomial $F \in K[x_0, x_1, \dots, x_n]$, $D(F) := \mathbb{P}^n \setminus V_P(F)$.

Theorem 1.10. *Let $W \subset \mathbb{P}^n$ be a closed projective variety. Let F be a homogeneous polynomial of degree d in $K[x_0, x_1, \dots, x_n]$ such that $W \nsubseteq V_P(F)$. Then $W \cap D(F)$ is an affine variety.*

Proof. The assumption $W \nsubseteq V_P(F)$ is equivalent to $W \cap D(F) \neq \emptyset$. Let us consider the d-tuple Veronese embedding $v_{n,d} : \mathbb{P}^n \to \mathbb{P}^{N(n,d)}$, with $N(n,d) = \binom{n+d}{d} - 1$, that gives the isomorphism $\mathbb{P}^n \simeq V_{n,d}$. In this isomorphism the hypersurface $V_P(F)$ corresponds to a hyperplane section $V_{n,d} \cap H$, for a suitable hyperplane H in $\mathbb{P}^{N(n,d)}$. Therefore we have $W \cap D(F) \simeq v_{n,d}(W \cap D(F)) = v_{n,d}(W) \setminus H = v_{n,d}(W) \cap (\mathbb{P}^{N(n,d)} \setminus H)$. There exists a projective isomorphism $\tau : \mathbb{P}^{N(n,d)} \to \mathbb{P}^{N(n,d)}$ such that $\tau(H) = H_0$, the fundamental hyperplane of equation $x_0 = 0$. Therefore, denoting $X := v_{n,d}(W)$, we get $X \cap (\mathbb{P}^{N(n,d)} \setminus H) \simeq \tau(X) \cap (\mathbb{P}^{N(n,d)} \setminus H_0) = \tau(X) \cap U_0$, which proves the theorem. $\square$

As a consequence of Theorem 1.10, we get that the open subsets of the form $W \cap D(F)$ form a topology basis composed of affine varieties for W.

Exercises 1.11. 1. Let $\varphi : \mathbb{A}^1 \to \mathbb{A}^n$ be the map defined by $t \to (t, t^2, \dots, t^n)$.

a) Prove that $\varphi : \mathbb{A}^1 \to \varphi(\mathbb{A}^1)$ is an isomorphism and describe $\varphi(\mathbb{A}^1)$;

b) give a description of φ^* and φ^{-1*}.

2. Prove that the Veronese variety $V_{n,d}$ is not contained in any hyperplane of $\mathbb{P}^{N(n,d)}$.

3. Let $GL_n(K)$ be the set of invertible $n \times n$ matrices with entries in K. Prove that $GL_n(K)$ can be given the structure of an affine variety.

4. Let $\varphi : X \to Y$ be a regular map and φ^* its comorphism. Prove that the kernel of φ^* is the ideal of $\varphi(X)$ in $\mathcal{O}(Y)$. In the affine case, deduce that φ is dominant if and only if φ^* is injective.

5. Prove that $\mathcal{O}(X_F)$ is isomorphic to $\mathcal{O}(X)_f$, where X is an affine algebraic variety, F a polynomial and f the regular function on X defined by F.