ESEMPI di P.I.

1) PARTICELLA LIBERA che si muove in $N = \mathbb{R}$ $(n=1)$

$$\underset{\underset{K_T(y_0, y_1)}{|||}}{\langle y_1 | e^{-HT/\hbar} | y_0 \rangle} \sim \left(\frac{m}{2\pi\hbar\Delta t}\right)^{1/2} \int \prod_{i=1}^{N-1} \left(\frac{m}{2\pi\hbar\Delta t}\right)^{1/2} dx_i \cdot e^{-\frac{m}{\hbar}\sum_{i=0}^{N-1}\frac{\Delta t}{2}\left(\frac{d(x_{i+1}, x_i)}{\Delta t}\right)^2}$$

$$\Delta t = \frac{T}{N}$$

$$= \left(\frac{m}{2\pi\hbar\Delta t}\right)^{1/2} \int \prod_{i=1}^{N-1} \frac{dx_i}{\left(\frac{2\pi\hbar\Delta t}{m}\right)^{1/2}} \, e^{-\frac{m}{2\hbar\Delta t}\left[(y_0 - x_1)^2 + (x_1 - x_2)^2 + \dots + (y_1 - x_{N-1})^2\right]}$$

Integrale Gaussiano:

$$\int_{-\infty}^{+\infty} dx \, e^{-a(x-\xi_1)^2 - b(x-\xi_2)^2} = \int_{-\infty}^{+\infty} dx \, e^{-(a+b)x^2 + 2x(a\xi_1 + b\xi_2) - a\xi_1^2 - b\xi_2^2}$$

$$= \int_{-\infty}^{+\infty} dx \, e^{-(a+b)\underbrace{\left[x^2 - 2x\left(\frac{a\xi_1 + b\xi_2}{a+b}\right) + \left(\frac{a\xi_1 + b\xi_2}{a+b}\right)^2\right]}_{\left[x - \frac{a\xi_1 + b\xi_2}{a+b}\right]^2} + \underbrace{\frac{(a\xi_1 + b\xi_2)^2}{a+b} - a\xi_1^2 - b\xi_2^2}}$$

$$\left[x - \frac{a\xi_1 + b\xi_2}{a+b}\right]^2, \quad \underset{x'}{\rotatebox{90}{=}} \left[\dots\right]$$

$$\frac{1}{a+b}\left\{ \cancel{a^2\xi_1^2} + \cancel{b^2\xi_2^2} + 2ab\xi_1\xi_2 - \cancel{a^2\xi_1^2} - ab\xi_1^2 - ab\xi_2^2 - \cancel{b^2\xi_2^2} \right\}$$

$$= -\frac{1}{a+b} ab (\xi_1 - \xi_2)^2$$

$$= \left\{ \int_{-\infty}^{+\infty} dx' \, e^{-(a+b)x'^2} \right\} e^{-\frac{ab}{a+b}(\xi_1 - \xi_2)^2}, \qquad \int_{-\infty}^{+\infty} dx' \, e^{-(a+b)x'^2} = \sqrt{\frac{\pi}{a+b}}$$

$$\int_{-\infty}^{+\infty} dx \, e^{-a(x-\xi_1)^2 - b(x-\xi_2)^2} = \sqrt{\frac{\pi}{a+b}} \, e^{-\frac{ab}{a+b}(\xi_1 - \xi_2)^2}$$

Iniziamo integrando in x_1

$$\int_{-\infty}^{+\infty} \frac{dx_1}{\left(\frac{2\pi\hbar\Delta t}{m}\right)^{1/2}} \, e^{-\frac{m}{2\hbar\Delta t}\left[(x_1 - y_0)^2 + (x_1 - x_2)^2\right]} = \left(\frac{\cancel{m}}{2\cancel{\pi}\cancel{\hbar}\cancel{\Delta t}}\right)^{1/2} \left(\frac{\cancel{2}\cancel{\hbar}\cancel{\Delta t}\cancel{\pi}}{2\cancel{m}}\right)^{1/2} e^{-\frac{m}{4\hbar\Delta t}(y_0 - x_2)^2} =$$

$$= \sqrt{\frac{1}{2}}\; e^{-\frac{m}{2\hbar\Delta t}\frac{1}{2}(x_2-y_0)^2}$$

Integriamo su x_2

$$\int_{-\infty}^{+\infty} \frac{dx_2}{\left(\frac{2\pi\hbar\Delta t}{m}\right)^{1/2}}\; e^{-\overset{a}{\frac{m}{4\hbar\Delta t}}(x_2-y_0)^2 - \overset{b}{\frac{m}{2\hbar\Delta t}}(x_2-x_3)^2} =$$

$$\frac{ab}{a+b} = \frac{2a^2}{3a} \qquad b=2a$$

$$= \left(\frac{m}{\cancel{2\pi\hbar\Delta t}}\right)^{1/2}\left(\frac{\overset{2}{\cancel{4}}\cancel{\pi\hbar}\cancel{\Delta t}}{3\cancel{m}}\right)^{1/2} e^{-\frac{m}{2\hbar\Delta t}\cdot\frac{1}{3}(y_0-x_3)^2}$$

$$= \sqrt{\frac{2}{3}}\; e^{-\frac{m}{2\hbar\Delta t}\frac{1}{3}(y_0-x_3)^2}$$

continuando con $\int dx_i \;\ldots$

$$\left(\frac{m}{2\pi\hbar\Delta t}\right)^{1/2}\int \prod_{i=1}^{N-1}\frac{dx_i}{\left(\frac{2\pi\hbar\Delta t}{m}\right)^{1/2}}\; e^{-\frac{m}{2\hbar\Delta t}\left[(y_0-x_1)^2+(x_1-x_2)^2+\ldots+(y_1-x_{N-1})^2\right]}$$

$$\Delta t = T/N$$

$$= \underbrace{\left(\frac{m}{2\pi\hbar\Delta t}\right)^{1/2}}_{=}\sqrt{\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\cancel{3}}\cdot\frac{\cancel{3}}{\cancel{4}}\cdots\frac{\cancel{N-1}}{N}}\; e^{-\frac{m}{2\hbar\Delta t}\frac{1}{N}(y_0-y_1)^2}$$

$$= \left(\frac{m}{2\pi\hbar T}\right)^{1/2}\sqrt{N}\cdot\sqrt{\frac{1}{N}}\; e^{-\frac{m\cdot N}{2\hbar T}\frac{1}{N}(y_0-y_1)^2}$$

$N\to\infty$

$$= \left(\frac{m}{2\pi\hbar T}\right)^{1/2} e^{-\frac{m}{2\hbar T}(y_0-y_1)^2}$$

Coi metodi usuali di QM: $\quad H = \frac{p^2}{2m}$

$$\langle y_1|\, e^{-\hat{H}T/\hbar}\,|y_0\rangle = \langle y_1|\, e^{-\hat{p}^2T/2m\hbar}\,\underset{\mathbb{1}}{\wedge}|y_0\rangle =$$

$$\mathbb{1} = \int\frac{dp}{2\pi\hbar}|p\rangle\langle p|$$

$$= \int\frac{dp}{2\pi\hbar}\langle y_1|e^{-\hat{p}^2T/2m\hbar}|p\rangle\langle p|y_0\rangle =$$

$$\langle p|y_0\rangle = e^{-ipy_0/\hbar}$$

$$= \int \frac{dp}{2\pi\hbar} e^{-p^2T/2m\hbar} e^{ip(y_1-y_0)/\hbar}$$

$$\left(\int_{-\infty}^{+\infty} dx\, e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}}\, e^{b^2/4a} \right.$$

$$= \frac{1}{2\pi\hbar} \left(\frac{2\pi\hbar m}{T}\right)^{1/2} e^{-\frac{(y_1-y_0)^2}{\hbar^2}\cdot\frac{2m\hbar}{4T}} = \left(\frac{m}{2\pi\hbar T}\right)^{1/2} e^{-\frac{m(y_1-y_0)^2}{2\hbar T}}$$

2) OSCILLATORE ARMONICO in $N=\mathbb{R}$

$$K_T(y_0,y_1) \equiv \langle y_1 | e^{-HT/\hbar} | y_0 \rangle = \int_{\substack{x(0)=y_0 \\ x(T)=y_1}} \mathcal{D}x\, e^{-S[x]/\hbar}$$

$$S^M_{o.a.} = \int dt' \underbrace{\left(\frac{m\dot{x}^2}{2} - \frac{m\omega^2 x^2}{2} \right)}_{L}$$

$$\Big\downarrow\ t' = -it$$

$$iS^M_{o.a.} = i(-i)\int dt \left(-\frac{m\dot{x}^2}{2} - \frac{m\omega^2}{2}x^2\right) = -\int dt \left(\frac{m\dot{x}^2}{2} + \frac{m\omega^2}{2}x^2\right)$$

$$= -S[x]$$

Facciamo un cambio di variabile di integrazione

$$x(t) = x_{cl}(t) + y(t)$$

dove $x_{cl}(t)$ la soluz. che minimizza S (Eucl.), cioè t.c.

$$\delta S[x_{cl}, \delta x] = 0 \quad \forall \delta x$$

$$\Leftrightarrow \quad m\ddot{x}_{cl} - m\omega^2 x_{cl} = 0 \quad \longrightarrow \text{ eq. del REPULSORE ARMONICO}$$

con $x_{cl}(0) = y_0$ e $x_{cl}(T) = y_1$

$$\Updownarrow$$

nuova variabile d'integrazione $y(t)$ è t.c. $y(0)=0$ e $y(T)=0$

$$K_T(y_0, y_1) = \int_{\substack{y(0)=0\\ y(T)=0}} \mathcal{D}y \; e^{-S[x_{cl}+y]/\hbar}$$

quadratica nel suo argomento

$\Rightarrow$ se espandiamo S attorno a un "pto" x_{cl} fino all'ordine quadratico, non trascuriamo alcun termine

$$S[x_{cl}+y] = S[x_{cl}] + \underbrace{\cancel{\frac{\delta S[x_{cl}]}{\delta x}}}_{=0} \cdot y + \frac{\delta^2 S}{\delta x^2} \cdot y^2$$

$$= S[x_{cl}] + S[y]$$

caso particolare di azione quadratica

$$K_T(y_0, y_1) = e^{-S[x_{cl}]/\hbar} \int_{\substack{y(0)=0\\ y(T)=0}} \mathcal{D}y \; e^{-\int_0^T \left(\frac{m}{2}\dot{y}^2 + \frac{m\omega^2}{2} y^2 \right) dt/\hbar}$$

integraz. per parti $\quad \dot{y}^2 = \frac{d}{dt}(y\dot{y}) - \underbrace{\ddot{y}y}_{}$ $\quad$ $=0$ agli estremi

$$= e^{-S[x_{cl}]/\hbar} \int_{\substack{y(0)=0\\ y(T)=0}} \mathcal{D}y \; e^{-\frac{\hbar}{2m}\int_0^T dt \; y\left(-\left(\frac{m}{\hbar}\right)^2 \frac{d^2}{dt^2} + \frac{m^2\omega^2}{\hbar^2} \right) y}$$

cambio variabile $\quad t = \frac{m}{\hbar} t'$

$$= e^{-S[x_{cl}]/\hbar} \int_{\substack{y(0)=0\\ y(T)=0}} \mathcal{D}y \; e^{-\frac{1}{2}\int_0^{T'} y\left(-\frac{d^2}{dt^2} + \left(\frac{m\omega}{\hbar}\right)^2 \right) y \, dt}$$

$T' = \frac{T\hbar}{m}$

funz. con $y(0) = y(T) = 0$

Cerchiamo autofunzioni dell'op. $-\frac{d^2}{dt^2}$ con $y(0) = y(T) = 0$

$$y_n = \sqrt{\frac{2}{T'}} \operatorname{sen}\left(\pi n \frac{t}{T'} \right) \longleftrightarrow \lambda_n = \left(\frac{\pi n}{T'}\right)^2 + \left(\frac{m\omega}{\hbar}\right)^2$$

base o.n. di autof. di $-\frac{d^2}{dt^2} + \left(\frac{m\omega}{\hbar}\right)^2$

$\downarrow$

$$y(t) = \sum_{n=1}^{\infty} \underset{\uparrow \text{ numeri}}{a_n} \sqrt{\frac{2}{T'}} \sin\left(\pi n \frac{t}{T'}\right)$$ ← Integrando su tutti i possibili valori delle a_n, integro su tutte le $y(t)$ t.c. $y(0)=y(T)=0$.

$$= e^{-S[x_{cl}]/\hbar}\, N \int \prod_{n=1}^{\infty} \frac{da_n}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\sum_{n=1}^{\infty} \lambda_n a_n^2}$$

$$= e^{-S[x_{cl}]/\hbar}\, N \prod_{n=1}^{\infty} \int \frac{da_n}{\sqrt{2\pi}}\, e^{-\lambda_n a_n^2/2}$$ ← Integrali Gaussiani

$$= e^{-S[x_{cl}]/\hbar}\, N \prod_{n=1}^{\infty} \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2\pi}{\lambda_n}} = e^{-S[x_{cl}]/\hbar} \frac{N}{\Big(\underbrace{\prod_{n=1}^{\infty} \lambda_n}_{\det}\Big)^{1/2}}$$

$$N \prod_{n=1}^{\infty} \lambda_n^{-1/2}$$

$$= e^{-S[x_{cl}]/\hbar} \frac{N}{\left(\det\left(-\frac{d^2}{dt^2} + \frac{m^2\omega^2}{\hbar^2}\right)\right)^{1/2}}$$

$$\prod_{n=1}^{\infty} \lambda_n^{-1/2} = \prod_{n=1}^{\infty} \left[\left(\frac{\pi n}{T'}\right)^2 + \left(\frac{m\omega}{\hbar}\right)^2\right]^{-1/2} =$$

$$= \left[\prod_{n=1}^{\infty} \left(\frac{\pi n}{T'}\right)^{-2}\right]^{1/2} \Bigg[\underbrace{\prod_{n=1}^{\infty} \frac{1}{1+\left(\frac{\pi n}{\omega T}\right)^{-2}}}_{\frac{\omega T}{\operatorname{senh}(\omega T)}}\Bigg]^{1/2} = \left[\frac{m\omega}{2\hbar \operatorname{senh}(\omega T)}\right]^{1/2}$$

regolarizzazione:

$$= \left[\frac{1}{2T'}\right]^{1/2} = \left[\frac{m}{2\hbar T}\right]^{1/2}$$

Γ ζ - function regularization : $\prod_{n=1}^{\infty} \left(\frac{2\pi n}{\beta} \right)^{-2}$ $\beta = 2T'$

Consideriamo la seguente serie

$$\zeta_1(s) = \sum_{n=1}^{\infty} \left(\frac{2\pi n}{\beta} \right)^{-2s}$$ $\leftarrow$ convergente per grande Re s e può essere continuata analiticamente vicino $s=0$

$$\zeta_1'(0) = \sum_{n=1}^{\infty} \log \left(\frac{2\pi n}{\beta} \right)^{-2} = \log \prod_{n=1}^{\infty} \left(\frac{2\pi n}{\beta} \right)^{-2}$$

$$\Rightarrow \prod_{n=1}^{\infty} \left(\frac{2\pi n}{\beta} \right)^{-2} = e^{\zeta_1'(0)}$$

$\zeta_1(s)$ è legata alla ζ di Riemann $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$

$$\zeta_1(s) = \left(\frac{\beta}{2\pi} \right)^{2s} \zeta(2s)$$

$$\zeta_1'(0) = 2 \log(\beta/2\pi) \underbrace{\zeta(0)}_{= -1/2} + 2 \underbrace{\zeta'(0)}_{-\frac{1}{2}\log 2\pi} = \log 1/\beta$$

$$\Rightarrow e^{\zeta_1'(0)} = 1/\beta$$

$\lrcorner$