$$K_T(y_0, y_1) = e^{-S[x_a]/\hbar} \frac{N}{\left(\det\left(-\frac{d^2}{dt^2} + \frac{m^2\omega^2}{\hbar^2}\right)\right)^{1/2}} = N\left(\frac{m\omega}{2\hbar \sinh(\omega T)}\right)^{1/2}$$

soluz. di eq.

$$\ddot{x}_a - \omega^2 x_a = 0 \quad \text{con} \quad \begin{aligned} x_a(0) &= y_0 \\ x_a(T) &= y_1 \end{aligned} \quad (\neq)$$

Calcoliamo $S[x_a]$.

eq. del repulsore armonico

$$x_a(t) = A e^{\omega t} + B e^{-\omega t}$$

Troviamo A e B t.c. $x_a(t)$ soddisfi le condiz. al bordo $(\neq)$

$$\begin{cases} y_0 = x_a(0) = A + B \\ y_1 = x_a(T) = A e^{\omega T} + B e^{-\omega T} \end{cases} \Leftrightarrow \begin{cases} y_1 - e^{-\omega T} y_0 = (e^{\omega T} - e^{-\omega T}) A \\ y_1 - e^{\omega T} y_0 = (e^{-\omega T} - e^{\omega T}) B \end{cases}$$

$$\Rightarrow \begin{cases} A = \dfrac{y_1 - e^{-\omega T} y_0}{e^{\omega T} - e^{-\omega T}} & \cdot\, e^{\omega t} \\ & + \\ B = \dfrac{y_1 - e^{\omega T} y_0}{e^{-\omega T} - e^{\omega T}} & \cdot\, e^{-\omega t} \end{cases}$$

$$\frac{e^x - e^{-x}}{2} = \sinh x$$

$$x_a(t) = \frac{y_1}{e^{\omega T} - e^{-\omega T}} \left(e^{\omega t} - e^{-\omega t}\right) - \frac{y_0}{e^{\omega T} - e^{-\omega T}} \left(e^{\omega t - \omega T} - e^{-\omega t + \omega T}\right)$$

$$x_a(t) = \frac{y_1}{\sinh(\omega T)} \sinh(\omega t) - \frac{y_0}{\sinh(\omega T)} \sinh(\omega t - \omega T)$$

$$\dot{x}_a(t) = \frac{y_1 \omega}{\sinh(\omega T)} \cosh(\omega t) - \frac{y_0 \omega}{\sinh(\omega T)} \cosh(\omega t - \omega T)$$

$$\mathcal{L}[x_{cl}] = \frac{m}{2}\left[\dot{x}_{cl}^2 + \omega^2 x_{cl}^2 \right] =$$

$$= \frac{m\omega^2}{2\,\mathrm{senh}^2(\omega T)} \Big\{ y_1^2 \Big[\underbrace{\mathrm{senh}^2(\omega t) + \cosh^2(\omega t)}_{\cosh(2\omega t)} \Big] + y_0^2 \Big[\mathrm{senh}^2(\omega t - \omega T) + \cosh^2(\omega t - \omega T) \Big]$$

$$- 2 y_0 y_1 \Big[\underbrace{\cosh(\omega t)\cosh(\omega t - \omega T) + \mathrm{senh}(\omega t)\,\mathrm{senh}(\omega t - \omega T)}_{\cosh(2\omega t - \omega T)} \Big]$$

$$S[x_{cl}] = \int_0^T dt \, \frac{m\omega^2}{2\,\mathrm{senh}^2(\omega T)} \Big\{ y_1^2 \cosh(2\omega t) + y_0^2 \cosh(2\omega t - 2\omega T) - 2 y_0 y_1 \cosh(2\omega t - \omega T) \Big\}$$

$$\mathrm{senh}\, 2x = 2\,\mathrm{senh}\, x \cosh x$$

$$= \frac{m\omega^{\cancel{2}}}{2\,\mathrm{senh}^{2}(\omega T)} \Big\{ \frac{y_1^2}{\cancel{2\omega}} \overbrace{\mathrm{senh}(2\omega T)}^{\cancel{2}\,\mathrm{senh}(\omega T)\cosh(\omega T)} + \frac{y_0^2}{\cancel{2\omega}} \mathrm{senh}(2\omega T) - \frac{2 y_0 y_1}{\cancel{2\omega}} \cdot \cancel{2}\, \underline{\mathrm{senh}(\omega T)}$$

$$S[x_{cl}] = \frac{m\omega}{2\,\mathrm{senh}(\omega T)} \Big\{ (y_0^2 + y_1^2)\cosh(\omega T) - 2 y_0 y_1 \Big\}$$

$$\lim_{\omega \to 0} S[x_{cl}] = \frac{m}{2T}\left(y_0^2 + y_1^2 - 2 y_0 y_1 \right) = \frac{m (y_0 - y_1)^2}{2T}$$

$$= S[x_{cl}]_{\text{particella libera}}$$

$$\parallel$$

$$\frac{m}{2}\left(\frac{y_0 - y_1}{T} \right)^2 \cdot T$$

$$K_T(y_0, y_1) = \mathcal{N}\left(\frac{m\omega}{2\hbar \sinh(\omega T)}\right)^{1/2} e^{-\frac{m\omega}{2\hbar\sinh(\omega T)}\left\{(y_0^2+y_1^2)\cosh(\omega T) - 2y_0y_1\right\}}$$

Fissiamo $\mathcal{N}$ richiedendo che $\lim_{\omega\to 0} K_T(y_0,y_1) = K_T^{\text{particella libera}}(y_0,y_1)$

$$\lim_{\omega\to 0} K_T(y_0,y_1) = \underbrace{\mathcal{N}\left(\frac{m}{2\hbar T}\right)^{1/2}}_{\text{dev'essere } \frac{m}{2\pi\hbar T}} e^{-\frac{m}{2\hbar T}(y_0-y_1)^2}$$

$$\Rightarrow \quad K_T(y_0,y_1) = \left(\frac{m\omega}{2\pi\hbar\sinh(\omega T)}\right)^{1/2} e^{-\frac{m\omega}{2\hbar\sinh(\omega T)}\left\{(y_0^2+y_1^2)\cosh(\omega T)-2y_0y_1\right\}}$$

FUNZIONE DI PARTIZIONE: $\int dy\, K_T(y,y) = \mathrm{Tr}\, e^{-\hat{H}T/\hbar}$

$$Z = \left(\frac{m\omega}{2\pi\hbar\sinh(\omega T)}\right)^{1/2} \int_{-\infty}^{+\infty} dy\, e^{-\frac{m\omega}{\hbar\sinh(\omega T)}\left\{\cosh(\omega T)-1\right\}y^2}$$

$$= \left(\frac{\cancel{m\omega}}{2\cancel{\pi}\cancel{\hbar}\,\cancel{\sinh(\omega T)}}\right)^{1/2}\left(\frac{\cancel{\pi}\,\cancel{\hbar}\,\cancel{\sinh(\omega T)}}{\cancel{m\omega}}\right)^{1/2} \frac{1}{\sqrt{\cosh(\omega T)-1}}$$

$$= \frac{1}{\sqrt{2}}\,\frac{1}{\sqrt{2\sinh^2(\omega T/2)\cancel{+1}\ \cancel{-1}}} = \frac{1}{2\sinh(\omega T/2)}$$

⇒ La funz. di partizione dell' oscillatore armonico (con $m=1$ $\omega=1$) è data da

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta \hat{H}} = \frac{1}{2 \sinh(\beta/2)}$$

Oscillatore armonico

Funz. di part. dell' osc. armonico col formalismo operatoriale

$$H = \frac{p^2}{2} + \frac{x^2}{2} = a^\dagger a + \frac{1}{2}$$

$$a = \frac{1}{\sqrt{2}}(p - ix) \qquad a^\dagger = \frac{1}{\sqrt{2}}(p + ix)$$

$$[a, a^\dagger] = 1 \qquad [H, a] = -a$$

$$[H, a^\dagger] = a^\dagger$$

$$H|\psi\rangle = E|\psi\rangle : \quad H(a|\psi\rangle) = (E-1)\, a|\psi\rangle$$

$$H(a^\dagger|\psi\rangle) = (E+1)\, a^\dagger|\psi\rangle \quad \rightsquigarrow \quad E_n = n + 1/2$$

$\rightsquigarrow$ ground state $|0\rangle$ s.t. $a|0\rangle = 0$ $\quad E_0 = 1/2$

$$\langle x|0\rangle = e^{-x^2/2} \qquad \left(\text{obbedisce } \left(-i\frac{d}{dx} - ix\right)\langle x|0\rangle = 0 \right)$$

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta H} = \sum_{n=0}^{\infty} e^{-\beta(n+1/2)} = \frac{1}{2\sinh(\beta/2)}$$

Applichiamo formula diretta dell'integrale di cammino per Z:

$$Z(\beta) = \int_{X(\beta)=X(0)} \mathcal{D}x \; e^{-\int_0^\beta dt \left(\frac{\dot{x}^2}{2} + \frac{x^2}{2} \right)}$$

azione Euclidea

$$= \int_{X(\beta)=X(0)} \mathcal{D}x \; e^{-\frac{1}{2}\int_0^\beta dt \; x \cdot \Theta x} \qquad \Theta = -\frac{d^2}{dt^2} + 1$$

Θ : autovalori λ_n	autofunzioni	deg.
1	cost.	1
$1+\left(\frac{2\pi n}{\beta}\right)^2$	$\begin{cases} \cos(2\pi n t/\beta) \\ \operatorname{sen}(2\pi n t/\beta) \end{cases}$	2

$$Z(\beta) = \frac{1}{\sqrt{\det\Theta}} = \prod_{\substack{\lambda \\ \text{autov.}}} \lambda^{-1/2}$$

$$= \prod_{n=1}^{\infty}\left[1+\left(\frac{2\pi n}{\beta}\right)^2\right]^{-1}$$

perché ogni autovalore compare con degenerazione 2

$$= \underbrace{\prod_{n=1}^{\infty}\left(\frac{2\pi n}{\beta}\right)^{-2}}_{\substack{\zeta\text{-function reg.} \\ = \frac{1}{\beta}}} \; \underbrace{\prod_{l=1}^{\infty}\frac{1}{1+\left(\frac{2\pi l}{\beta}\right)^{-2}}}_{\frac{\beta}{2\operatorname{senh}(\beta/2)}}$$

$$\Downarrow$$

$$Z(\beta) = \frac{1}{2\operatorname{senh}(\beta/2)}$$

3) PARTICELLA LIBERA che vive su $N = S^1_R$ ← cerchio di raggio $R/2\pi$

$(m=1)$ $\qquad$ $x \cong x + R$

$$S = \int \frac{1}{2} \dot{x}^2 \, dt \qquad H = \frac{p^2}{2} = -\frac{1}{2} \frac{d^2}{dx^2}$$

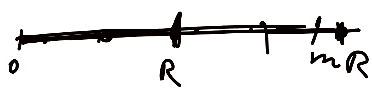
$$H \to \quad \psi_n = \frac{1}{\sqrt{R}} e^{2\pi i n x/R} \qquad E_n = \frac{2\pi^2 n^2}{R^2} \qquad n \in \mathbb{Z}$$

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta \hat{H}} = \sum_{n=-\infty}^{+\infty} e^{-\beta 2\pi^2 n^2/R^2}$$

Calcoliamo funz. di part. col P.I.

$$Z(\beta) = \int_{\mathcal{C}} Dx \, e^{-S[x]}$$

mappe da S^1_T a S^1_R



Queste mappe cadono in diverse classi topologiche, indicizzate dal WINDING NUMBER $m \in \mathbb{Z}$

di volte in cui S^1_T è arrotolato attorno a S^1_R

$\mathcal{C}$ è disconnesso

$\mathcal{C} = \bigcup_m \mathcal{C}_m$

[In ognuna di queste classi c'è una soluz. delle eq. del moto classiche rispetto alla top. euclidea]

$$Z(\beta) = \sum_{m=-\infty}^{+\infty} \int_{\mathcal{C}_m} Dx_m \, e^{-S[x_m]}$$

condiz. al bordo per mappe in C_m

$$x_m(\beta) = x(0) + mR$$

$$\Downarrow$$

$$x_m(t) = x_0(t) + \frac{mtR}{\beta}$$

funz. veram. periodiche in t

$$S = \int \frac{1}{2} \dot{x}^2 \, dt$$

$$S[x_m] = \frac{m^2R^2}{2\beta} + \int_0^\beta x_0 \left(-\frac{1}{2}\frac{d^2}{dt^2}\right) x_0 \, dt$$

$$Z = \sum_{m=-\infty}^{+\infty} e^{-\frac{m^2R^2}{2\beta}} \underbrace{\int \mathcal{D}x_0 \, e^{-\int_0^\beta x_0 \left(-\frac{1}{2}\frac{d^2}{dt^2}\right) x_0 \, dt}}$$

$$\frac{1}{\sqrt{\det\left(-\frac{1}{2}\frac{d^2}{dt^2}\right)}} \rightsquigarrow \frac{R\sqrt{\beta}}{\sqrt{2\pi}} \frac{1}{\sqrt{\det'\left(-\frac{1}{2}\frac{d^2}{dt^2}\right)}}$$

ha un autovalore NULLO ($\sim$ zero modo)

det esclusi gli zero modi

$$\mathcal{D}x_0 \rightsquigarrow \int \prod_n \frac{dc_n}{\sqrt{2\pi}} e^{-\lambda_n c_n^2}$$

zero-modo

$$\int_0^{R\sqrt{\beta}} \frac{dc_0}{\sqrt{2\pi}} \cdot 1 = \frac{R\sqrt{\beta}}{\sqrt{2\pi}}$$

$$\frac{1}{\sqrt{\det'\left(-\frac{1}{2}\frac{d^2}{dt^2}\right)}} \longrightarrow \qquad \det'\left(-\frac{d^2}{dt^2}\right) = \prod_{m\neq 0}\left(\frac{2\pi m}{\beta}\right)^2 = \beta^2$$

$$Z = \frac{R}{\sqrt{2\pi\beta}} \sum_{m=-\infty}^{+\infty} e^{-\frac{m^2R^2}{2\beta}}$$

$$=? \qquad \sum_{n=-\infty}^{+\infty} e^{-\beta\, 2\pi^2 n^2/R^2}$$