Expected utility and mixed strategies

In order to evaluated a player's strategy against a strategy profile played by others we need to compute its expected value.
For example for player i, the expected utility of strategy p_i against the others' strategies p_{-i} is denoted by $u_i(p_i, p_{-i})$ or $E_i(p_i|p_{-i})$

Example:

		Player 2		
		L	C	R
0.4	T	2,1	2,2	5,0
Player 1 0.3	Y	3,2	5,3	3,1
0.2	Z	4,3	1,1	2,2
0.1	B	1,4	0,1	4,4

We compute the expected value of $p_2 = (1, 0, 0)$ (the pure strategy L) against the player 1's strategy $p_1 = (0.4, 0.3, 0.2, 0.1)$

$$E_2(p_2|p_1) = 1 \cdot 0.4 + 2 \cdot 0.3 + 3 \cdot 0.2 + 4 \cdot 0.1 = 2$$

Example:

		Player 2		
		L	C	R
Player 1	T	2,1	2,2	5,3
	Y	3,2	5,3	3,1
	Z	4,3	1,1	2,2
	B	1,4	0,1	4,5

We compute the expected value of $p_2 = (0.6, 0, 0.4)$ (a mixed strategy L) against the player 1's strategy $p_1 = (0.3, 0, 0, 0.7)$

		Player 2		
		0.6	**0**	**0.4**
Player 1	**0.3**	2,1	2,2	5,3
	0	3,2	5,3	3,1
	0	4,3	1,1	2,2
	0.7	1,4	0,1	4,5

$$E(p_2|p_1) = 1 \cdot 0.6 \cdot 0.3 + 3 \cdot 0.4 \cdot 0.3 + 4 \cdot 0.6 \cdot 0.7 + 5 \cdot 0.4 \cdot 0.7$$

Sometime, when we take the expectation of a pure strategy, inside the parenthesis we write the name of the pure strategy.

For example in the game

		Player 2		
		L	C	R
Player 1	T	2,1	2,2	5,3
	Y	3,2	5,3	3,1
	Z	4,3	1,1	2,2
	B	1,4	0,1	4,5

the expected value of $p_2 = (0, 1, 0)$, i.e. the pure strategy C, can be denoted by either $E_2(p_2|p_1)$ or $E_2(C|p_1)$.

Finally if it is not ambiguous we omit the strategy of the opponent(s), e.g. in the above example we write $E_2(C)$

		Player 2		
		L	C	R
Player 1	T	2,1	2,2	5,3
	Y	2,2	4,3	3,1
	Z	4,3	1,1	2,2
	B	1,4	0,1	4,5

Compute the expected value of $p_1 = (0, 1, 0, 0)$ against $p_2 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

$$E_1(p_1|p_2) = 2 \cdot \frac{1}{3} + 4 \cdot \frac{1}{3} + 3 \cdot \frac{1}{3} = 3$$

		Player 2		
		L	C	R
Player 1	T	2,1	2,2	5,3
	Y	2,2	4,3	3,1
	Z	4,3	1,1	2,2
	B	1,4	0,1	4,5

Question 1

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Compute the expected value of $p_1 = (\frac{1}{2}, \frac{1}{2}, 0, 0)$ against $p_2 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

$$E_1(p_1|p_2) =$$

$$= 2 \cdot \frac{1}{3} \cdot \frac{1}{2} + 2 \cdot \frac{1}{3} \cdot \frac{1}{2} + 5 \cdot \frac{1}{3} \cdot \frac{1}{2} + 2 \cdot \frac{1}{3} \cdot \frac{1}{2} + 4 \cdot \frac{1}{3} \cdot \frac{1}{2} + 3 \cdot \frac{1}{3} \cdot \frac{1}{2} =$$

$$= 2 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} = 3$$

Matching Pennies

		Player 2	
		Head (q)	Tail ($1-q$)
Player 1	Head (r)	1,-1	-1,1
	Tail ($1-r$)	-1,1	1,-1

$p_1 = (r, 1-r)$ *where r is the probability that player 1 chooses Head,*

$p_2 = (q, 1-q)$ *where q is the probability that player 2 chooses Head*

Player 1's expected payoff is:

$$E_1(p_1|p_2) =$$

$$= rq - r(1-q) - (1-r)q + (1-r)(1-q) =$$

$$= r(4q-2) + 1 - 2q$$

Player 1's expected payoff is:

$$E_1(p_1|p_2) = r(4q - 2) + 1 - 2q$$

The expected payoff:

1) is increasing in r if *(4 q – 2)>0 i.e. q>0.5*

   In this case the best response of player 1 is $p_1 = (1,0)$

2) is decreasing in r if *(4 q – 2)<0 i.e. q<0.5*

   In this case the best response of player 1 is $p_1 = (0,1)$

3) is equal *0* and constant in r for *q = 0.5*

   In this case the best response of player 1 is

$$p_1 = (r, 1 - r)\ \forall r \ \in [0, 1]$$

		Player 2	
		Head (q)	Tail
Player 1	Head (r)	1,-1	-1,1
	Tail	-1,1	1,-1

- *r:* Probability that 1 chooses Head
- *q:* Probability that 2 chooses Head

$r(q) =$ 1 if $q > 1/2$;

0 if $q < 1/2$;

[0,1] if $q = 1/2$

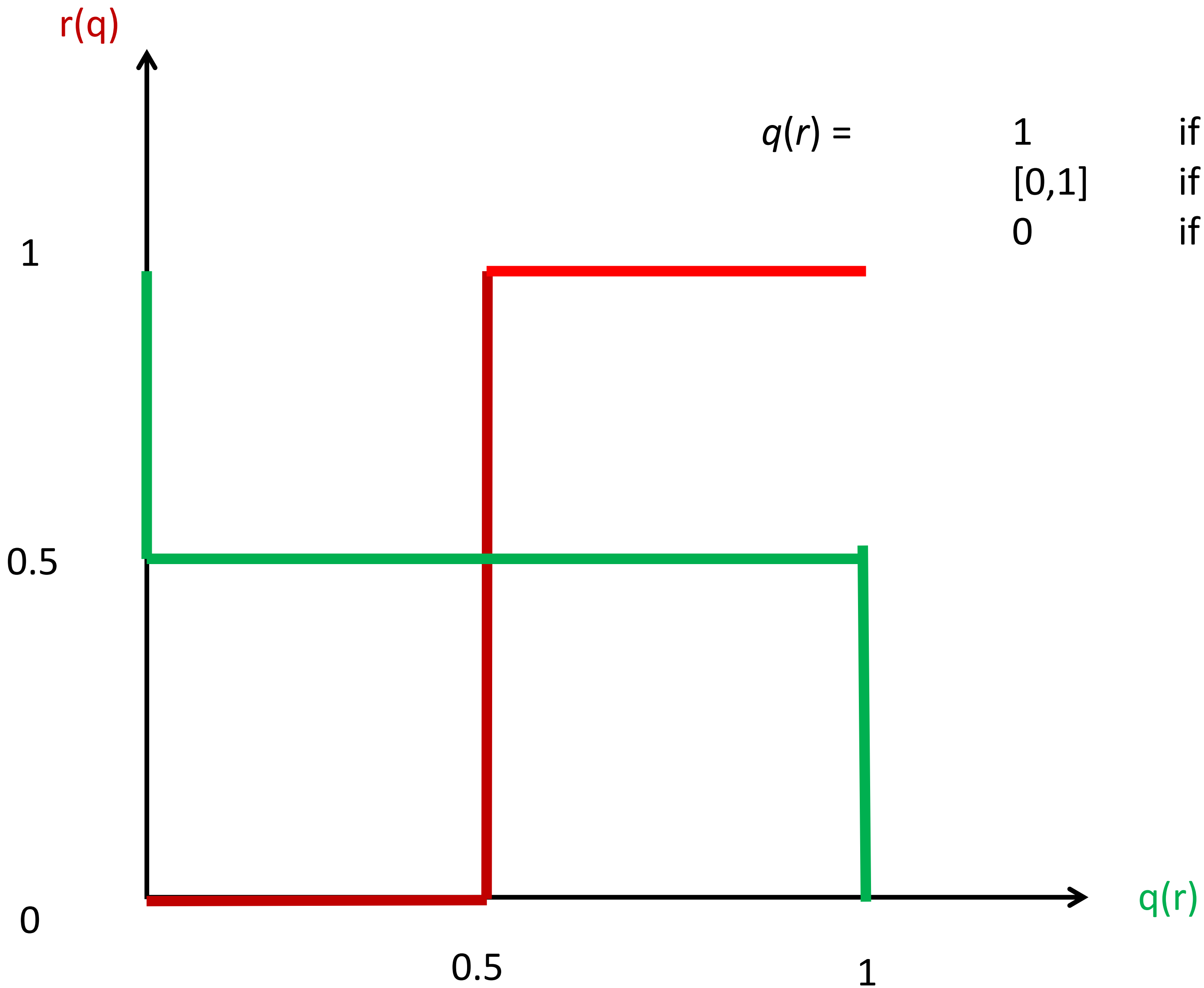
$q(r) =$ 0 if $r > 1/2$;

1 if $r < 1/2$

[0,1] if $r = 1/2$

$$r(q) = \begin{cases} 0 & \text{if } q < 1/2 \\ [0,1] & \text{if } q = 1/2 \\ 1 & \text{if } q > 1/2; \end{cases}$$

$$q(r) = \begin{cases} 1 & \text{if } r < 1/2 \\ [0,1] & \text{if } r = 1/2 \\ 0 & \text{if } r > 1/2; \end{cases}$$



Note that player 1's strategy (0.5, 0.5) is a best response to the player 2' strategy (0.5, 0.5) and
player 2's strategy (0.5, 0.5) is a best response to the player 1's strategy (0.5, 0.5)
Then player 1 plays (0.5, 0.5) and player 2 plays (0.5, 0.5) is a Nash equilibrium in mixed strategies

Definition of Nash equilibrium with mixed strategies:
In a normal form game $G = (S_1, \ldots S_n; u_1, \ldots, u_n\}$ the mixed strategy profile $(p_1^*, \ldots p_n^*)$ is a Nash equilibrium if each player's mixed strategy is a best response to the other players' strategies.

Battle of the Sexes

		Player 2	
		Ball (q)	Theatre
Player 1	Ball (r)	2,1	0,0
	Theatre	0,0	1,2

$p_1 = (r, 1 - r)$ *where r is the probability that player 1 chooses Ball*

$p_2 = (q, 1 - q)$ *where q is the probability that player 2 chooses Ball*

Player 1's expected payoff is:

$E_1(p_1) = 2\,r\,q + (1 - r)(1 - q) = r\,(3q - 1) + 1 - q$

It is increasing in r if $(3\,q - 1) > 0$ *i.e.* $q > 1/3$ $\rightarrow BR_1$ *is* $(1, 0)$

It is decreasing in r if $(3\,q - 1) < 0$ *i.e.* $q < 1/3$ $\rightarrow BR_1$ *is* $(0, 1)$

It is constant and equal $2/3$ for $q = 1/3$ $\rightarrow BR_1$ *is* $(r, 1 - r)\forall r \in [0,1]$

		Player 2	
		Ball	Theatre
Player 1	Ball	2,1	0,0
	Theatre	0,0	1,2

Consider player 2

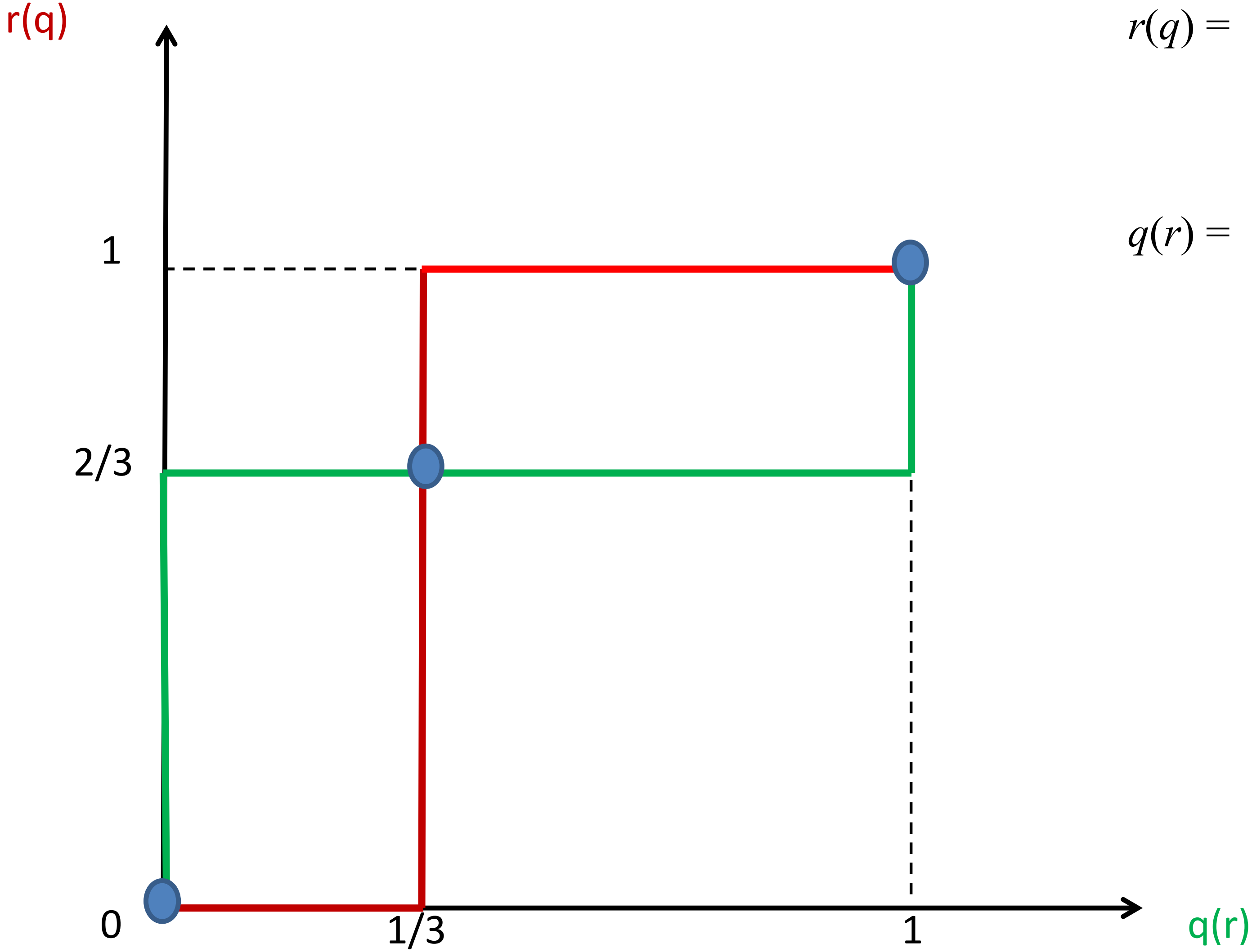
$E_2(p_2) = 1\,q\,r + 2\,(1-q)\,(1-r) = q(3\,r - 2) + 2 - 2\,r$

It is increasing in *q* if *(3r –2)>0 i.e. r>2/3* → BR_2 *is (1, 0)*

It is decreasing in *q* if *(3r–2)<0 i.e. r<2/3* → BR_2 *is (0, 1)*

It is constant and equal *2/3* for *r=2/3* → BR_2 *is* $(q, 1-q)\forall q \in [0,1]$

$$r(q) = \begin{cases} 1 & \text{if } q > 1/3; \\ 0 & \text{if } q < 1/3; \\ [0,1] & \text{if } q = 1/3 \end{cases}
\qquad
q(r) = \begin{cases} 1 & \text{if } r > 2/3; \\ 0 & \text{if } r < 2/3; \\ [0,1] & \text{if } r = 2/3 \end{cases}$$



$$r(q) = \begin{cases} 1 & \text{if } q > 1/3; \\ 0 & \text{if } q < 1/3; \\ [0,1] & \text{if } q = 1/3 \end{cases}$$

$$q(r) = \begin{cases} 1 & \text{if } r > 2/3; \\ 0 & \text{if } r < 2/3; \\ [0,1] & \text{if } r = 2/3 \end{cases}$$

Equilibria:

$(r, q) = (1,1) =$ (Ball, Ball),

$(r, q) = (0,0) =$ (Theatre,Theatre)

$(r, q) = (2/3,1/3)$

$$p_1 = (1, 0), p_2 = (1, 0)$$
$$p_1 = (0, 1), p_2 = (0, 1)$$
$$p_1 = (2/3,1/3), p_2 = (1/3,2/3)$$

Characterization of mixed-strategy Nash equilibria

Proposition:

$p^* = (p_1^*, \ldots p_n^*)$ is a mixed-strategy Nash equilibrium **if and only if** the following conditions are satisfied:

1) each action s_j that is played by player *i* with strictly positive probability (according to p_i^*) yields **the same expected payoff** to *i* as strategy p_i^*
2) every action s_j' that is played by *i* with probability 0 (according to p_i^*) yields **at most the same expected payoff** to *i* as strategy p_i^*

assuming, in both cases, that other players play as predicted in the Nash equilibrium $(p_1^*, \ldots p_n^*)$

Useful tips for finding mixed-strategy Nash equilibria

Step 1: Consider player *i,* take a subsets S_i' of its strategies and assume that only these strategies are played by a strictly positive probability

Step 2: Look for the other players' strategies that allow to satisfy conditions 1) and 2), i.e.

a) The expected payoffs to play each one of the strategies in S_i' are equal to each other:

$$E_i(s_j) = E_i(s_w)\ \forall s_j, s_w \in S_i'$$

b) The expected payoffs to play each one of the strategies that are not in S_i' are not greater than the expected payoff of the strategies in S_i':

$$E_i(s_j) \leq E_i(s_w)\ \forall s_j \in S_i\ /S_i', s_w \in S_i'$$

Step 3: check if the other players' strategies you have found in step 2 satisfy conditions 1 and 2

Step 4: Repeat this procedure for all possible strategies' subsets of player *i*

		Player 2	
		L	R
Player 1	T	2,3	5,0
	M	3,2	1,4
	B	1,5	4,1

No equilibrium in pure strategies.

There is no equilibrium where player 1 chooses B with strictly positive probability. T strictly dominates B, so whatever player 2 does, player 1 can increase its expected payoff by playing T instead of B. Then $p_{1B} = 0$.

That leaves player 1 choosing among T and M.

		Player 2	
		L (by l)	R (by $1-l$)
Player 1	T (by prob t)	2,3	5,0
	M (by $1-t$)	3,2	1,4
	~~B~~	~~1,5~~	~~4,1~~

That leaves player 1 choosing among T and M.

Let be $p_{1T} = t$ and $p_{2L} = l$

To play T and M, both with strictly positive probability requires:

$$E_1(T) = E_1(M) \rightarrow 2l + 5(1-l) = 3l + 1(1-l) \rightarrow l = 4/5$$

To play L and R, both with strictly positive probability requires:

$$E_2(L) = E_2(R) \rightarrow 3t + 2(1-t) = 4(1-t), \rightarrow t = 2/5$$

Nash Equilibrium:

$$((p_{1T}, p_{1M}, p_{1B}), (p_{2L}, p_{2R})) = ((2/5, 3/5, 0), (4/5, 1/5))$$

		Player 2	
		L (by l)	R (by $1-l$)
Player 1	T (by prob t)	2,3	1,0
	M (by $1-t$)	1,2	3,4

Question 2

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Compute p_{2L} in the mixed Nash equilibrium

Let be $p_{1T} = t$ and $p_{2L} = l$

To play T and M, both with strictly positive probability requires:

$$E_1(T) = E_1(M) \rightarrow 2l + 1(1 - l) = 1l + 3(1 - l) \rightarrow l = 2/3$$

To play L and R, both with strictly positive probability requires:

$$E_2(L) = E_2(R) \rightarrow 3t + 2(1 - t) = 4(1 - t), \rightarrow t = 2/5$$