STOCHASTIC APPROXIMATION

- MEAN FIELD
- APPROXIMATE STOCHASTIC MODELS
- MOMENT CLOSURE

MEAN FIELD

(X, S, x_0, R) ,

$X : X_1, \dots, X_n$

S = state space

$x_0 \in S$

$R \ni \rho \quad \rho: r_1 X_1 + \dots + r_n X_n \to p_1 X_1 + \dots + p_n X_n$

$\cdot \; v_\rho = p_\rho - r_\rho \quad [X \to X + v_\rho]$

$f_\rho(x) \geq 0$ rate function / rate in state x of ρ

SYSTEM SIZE: TOTAL POPULATION
VOLUME (with constant density)
AREA

N = system size

population DENSITY

$$\hat{X} = \frac{X}{N}$$

SIR $\{S, I, R\}$

$$\hat{X}_S = \frac{X_S}{N}$$

$N : X_S + X_I + X_R$

or

$N : X_S(0) + X_I(0) + X_R(0)$

$\mathcal{M} = (X, S, x_0, R)$ POPULATION MODEL. SYSTEM SIZE N.

$$\hat{\mathcal{M}} = (\hat{X}, \hat{S}, \hat{x}_0, \hat{R})$$

$$\hat{X}^{(N)} = \frac{X}{N} \; ; \; \hat{S}^{(N)} = \left\{ \hat{s} = \frac{s}{N} \;\middle|\; s \in S \right\} ; \; \hat{x}_0^{(N)} = \frac{x_0}{N}$$

$\hat{R} \ni \hat{\rho}$ for $\rho \in R$

$$\hat{\rho} : \frac{r_1}{N}\hat{X}_1 + \dots + \frac{r_n}{N}\hat{X}_n \to \frac{p_1}{N}\hat{X}_1 + \dots + \frac{p_n}{N}X_n$$

$$\Rightarrow \hat{v}_{\hat{\rho}}^{(N)} = \frac{v_\rho}{N}$$

$$f_\rho(x) = \boxed{N \hat{f}_{\hat{\rho}}(\hat{x})}$$

$$\hat{f}_{\hat{\rho}}^{(N)}(\hat{x}) := \frac{f_\rho(N\hat{x})}{N}$$ on

N: system size

$$\mathcal{M} = (X, S, x_0, R) \rightsquigarrow \underbrace{\hat{\mathcal{M}}^{(N)} = (\hat{X}^{(N)}, \hat{S}^{(N)}, \hat{x}_0^{(N)}, \hat{R}^{(N)})}$$

MEAN FIELD / FLUID APPROXIMATION : study $\mathcal{M}^{(N)}$ as $N \to \infty$

(SIR $\hat{X}_S^{(N)} \in \{0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N}{N}\} \subseteq [0,1]$)

$$\underbrace{\hat{S}^{(N)} \subseteq E \subseteq \mathbb{R}^n, \forall N}$$

$\hat{X}^{(N)}(t)$ CTMC for normalized model. How does it look like as $N \to \infty$

$$\frac{d\hat{x}^{(N)}}{dt} = \underbrace{F^{(N)}(\hat{x}^{(N)})}$$ drift : expected increment in $\hat{x}^{(N)}$ at time t.

$$:= \sum_{\rho \in R} \hat{v}_\rho \cancel{N} \hat{f}_\rho(\hat{x}^{(N)}) = \boxed{\sum_{\rho \in R} v_\rho \cdot \hat{f}_\rho(\hat{x}^{(N)})}$$

$\hookrightarrow \frac{v_\rho}{\cancel{N}}$

REQUIREMENT: $\hat{f}^{(N)}_{\hat{\jmath}}(\hat{x}) \longrightarrow \hat{f}_{\hat{\jmath}}(\hat{x})$ defined on E uniformly on $\hat{x}$

$\Big(\underbrace{\| \hat{f}^{(N)}_{\hat{\jmath}}(\hat{x}) - \hat{f}_{\hat{\jmath}}(\hat{x}) \|_\infty}_{} \to 0 \Big)$

$$\Rightarrow \quad F^{(N)}(\hat{x}) \xrightarrow{N\to\infty} F(\hat{x}) := \sum_{\rho \in R} v_\rho \cdot \hat{f}_{\hat{\rho}}(\hat{x})$$

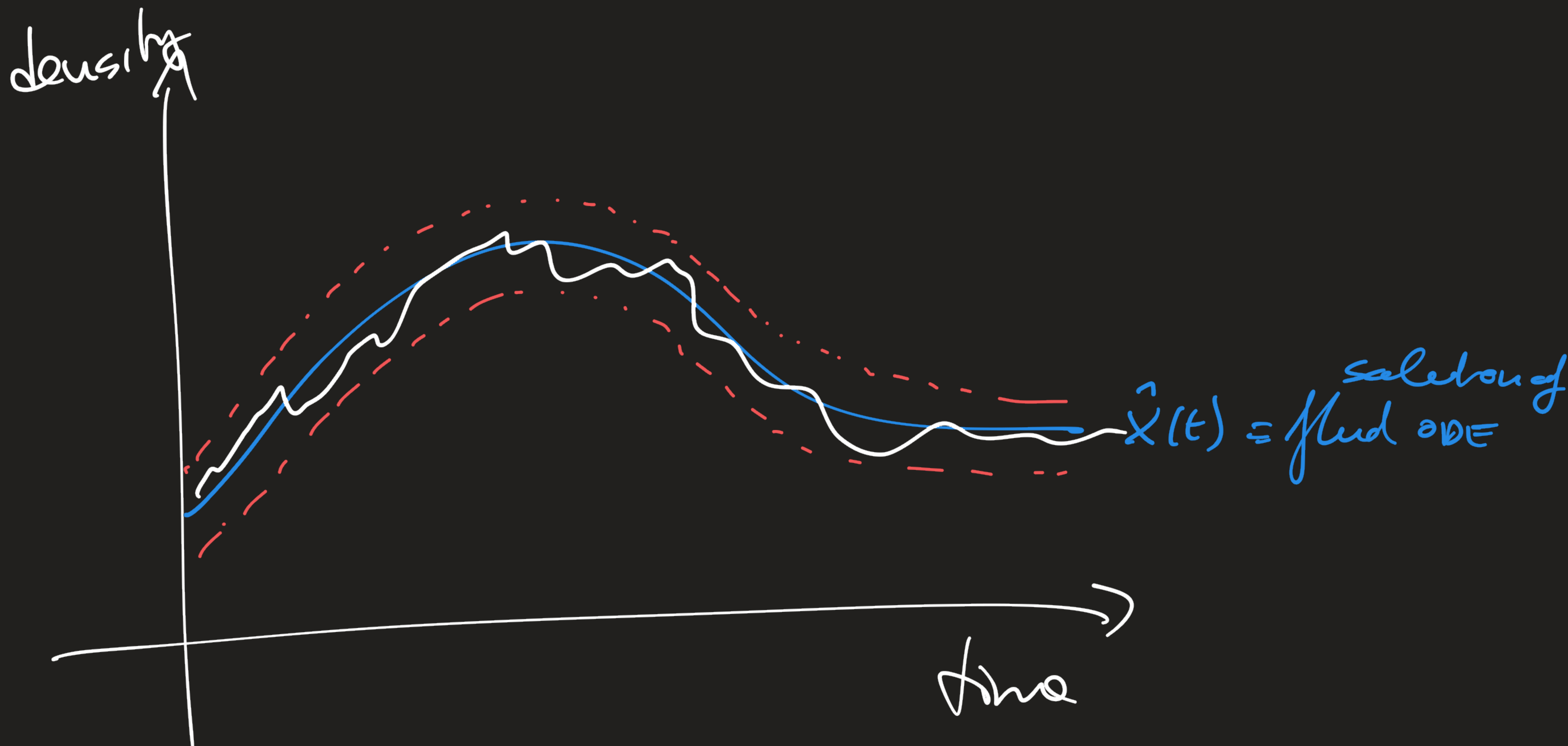
CONSTRUCT $\Rightarrow$ $\boxed{\dfrac{d\hat{x}}{dt} = F(\hat{x})}$ mean field / fluid ODE

with $\hat{x}(0) = \hat{x}_0 \xleftarrow{N\to\infty} \hat{x}_0^{(N)}$

THEOREM: if F is Lipschitz (locally), then for $\underline{T<\infty}$,

$$\lim_{N\to\infty} \sup_{t\le T} \| \hat{x}^{(N)}(t) - \hat{x}(t) \| = 0 \quad \text{(in probability)}$$

$$\left(\text{i.e.} \quad \lim_{N\to\infty} P\left(\sup_{t\le T} \| \hat{x}^{(N)}(t) - \hat{x}(t) \| > \varepsilon \right) = 0, \ \forall \varepsilon > 0 \right)$$



FOR large N, you can "replace" the stoch. system with fluid ODE. (for TRANSIENT)

SIR $\quad X_S, X_I, X_R \quad S \in \mathbb{N}^3$

$$S + I \to S + S \quad , \; k_I \frac{X_S \cdot X_I}{N}$$

$$I \to R \quad , \; k_R X_I$$

$$R \to S \quad , \; k_S \cdot X_R$$

$$X_S + X_I + X_R = N \quad \text{TOTAL POP.}$$

$$\hat{X}_S, \hat{X}_I, \hat{X}_R := \frac{X_S}{N}, \frac{X_I}{N}, \frac{X_R}{N}$$

- Recovery $\quad k_R X_I = N k_R \frac{X_I}{N} = N \underbrace{k_R \hat{X}_I^{(N)}}_{f_R(\hat{X}) = k_R \hat{X}_I^{(N)}}$

- Infection $\quad N \cdot \frac{k_I X_S X_I}{N \, N} = N \underbrace{k_I \hat{X}_I \hat{X}_S}_{f(\hat{X}) = k_I \hat{X}_I \hat{X}_S}$

LINEAR NOISE

$$\mathcal{M}^{(N)} = \left(X^{(N)}, S^{(N)}, x_0^{(N)}, R^{(N)}\right) \rightsquigarrow \underbrace{\hat{\mathcal{M}}^{(N)} = \left(\hat{X}^{(N)}, \hat{S}^{(N)}, \hat{x}_0^{(N)}, \hat{R}^{(N)}\right)}$$

- MICROSCOPIC $\Longleftrightarrow$ we need full stochastic description (small population) CTMC
- MESOSCOPIC $\Longleftrightarrow$ stochastic dynamics, but no need to describe individuals LINEAR NOISE
- MACROSCOPIC $\Longleftrightarrow$ large pop. limit : deterministic description MEAN FIELD

ANSATZ: $\underbrace{\hat{X}^{(N)}(t) \approx \underbrace{\hat{x}(t)}_{\text{determ}} + N^{-1/2}\underbrace{\xi(t)}_{\text{random}}}$ ← ξ does <u>not</u> depend on N

$$\frac{d}{dt}P\left(\hat{X}^{(N)},t\right) = \sum_{j\in R^{(N)}} N\hat{f}_j\left(\hat{X}^{(N)}-\hat{v}_j\right)p\left(\hat{X}^{(N)}-\hat{v}_j,t\right) - \sum_{j\in R^{(N)}} N\hat{f}_j\left(\hat{X}^{(N)}\right)\cdot p\left(\hat{X}^{(N)},t\right)$$

- APPLY ANSATZ
- EXPAND $P\left(\underbrace{\hat{x}+N^{-1/2}\xi},t\right)$ around $\hat{x}$, reasonable for $N^{-1/2}\xi$ is small
- COLLECT TERMS FOR $\Pi(\xi,t)$ (density of ξ at time t)

$$F(\hat{x}) = \sum_{\rho} v_\rho \cdot \hat{f}_\rho(\hat{x}) \quad \text{DRIFT}$$

$$J_{\rho}(t) = (J_{ij}) \qquad J_{ij}(t) = \sum_{\rho} v_\rho[i]\, \partial_j f_\rho(\overbrace{\hat{x}}^{\text{MF solution}}(t))$$

$$D(\hat{x}) = \left(D_{ik}(\hat{x})\right)_{ik} \qquad D_{ik}(\hat{x}) = \sum_{\rho} v_\rho[i]\, v_\rho[k]\, \hat{f}_\rho(\hat{x})$$

$\rightsquigarrow \xi(t) \in \mathbb{R}^n$ CONTINUOUS QUANTITY

$$\frac{\partial \Pi(\xi, t)}{\partial t} = \sum_{i,j} \underbrace{J_{ij}(t)}_{\text{on MF}}\, \partial_i \left(\xi_j \Pi(\xi, t)\right) + \frac{1}{2} \sum_{i,j} \underbrace{D_{ij}}_{\substack{D_{ij}(\hat{x}(t)) \\ \text{on MF.}}}\, \partial_{ij} \Pi(\xi, t)$$

$\Downarrow$

The solution $p(\xi(t))$ is GAUSSIAN distribution on

$\mu(t) = \mathbb{E}[\xi(t)] =$

$C(t) = \mathrm{cov}[\xi(t)]$

$\int \xi \, \pi(\xi, t) \, d\xi = \mu(t)$ $\qquad \frac{d}{dt}\mu(t) = \int \xi \underbrace{\frac{d}{dt}\pi(\xi,t)}\, d\xi$

- $\frac{d}{dt}\mu(t) = J(t) \cdot \mu(t)$ $\quad$. (if $\mu(0) = 0 \Rightarrow \mu(t) \equiv 0 \ \forall t$)

- $\frac{d}{dt} C(t) = J(t) \cdot C(t) + C(t) \cdot J^{T}(t) + D(t)$

- Solve the MF equation $\hat{x}(t)$
- Solve the equation for $C(t)$ ($O(n^2)$ equations)

Then at time t, $\mathcal{N}(\hat{x}(t), \frac{1}{N} C(t))$

MOMENT CLOSURES

- FIND ODEs FOR MOMENTS of a stochastic process

$$\mathbb{E}_t[X_j^m] \qquad \mathbb{E}_t[X_1 X_2] \qquad \mathbb{E}_t[X_1^2]$$

we focus on NON-CENTERED MOMENTS, like

$$\mathbb{E}_t[X_j^2]$$

in contrast with CENTERED moments

$$\mathbb{E}_t\left[(X_j - \mathbb{E}[X_j])^2\right]$$

$$\mathbb{E}_t[X_j^2] = \sum_{x \in S} x_j^2 \cdot \underbrace{p(x,t)}_{\text{solution of CME}}$$

$$\frac{d\,\mathbb{E}_t\left[X_1^{m_1}\cdots X_n^{m_n}\right]}{dt} = \sum_{\rho\in R} \mathbb{E}_t\left[f_\rho(\mathbf{X}) \left(\prod_{j=1}^{n} (X_j + v_\rho[j])^{m_j} - X_1^{m_1}\cdots X_n^{m_n} \right) \right]$$

Dynkin FORMULA

$$\frac{d\,\mathbb{E}_t[X]}{dt} = \mathbb{E}_t\left[F(\mathbf{X})\right] = \mathbb{E}_t\left[\sum_{\rho\in R} v_\rho \cdot f_\rho(X) \right]$$

- IF $f_\rho(X)$ is a linear function of X, then $\mathbb{E}_t[f_\rho(X)] = f_\rho(\mathbb{E}[X])$
- IF $f_\rho(x)$ is non-linear, then $\mathbb{E}_t[f_\rho(x)] \neq \underbrace{f_\rho(\mathbb{E}[X])}_{\text{is a FIRST ORDER APPROX of } \mathbb{E}_t[f_\rho(X)]}$

$$\frac{d\mathbb{E}_t[X]}{dt} = F(\mathbb{E}_t[X])$$ which is the MEAN FIELD equation

$$\frac{d\mathbb{E}_t[X_I]}{dt} = k_I \underbrace{E_t[X_I X_S]}_{E_t[X_I]\,E_t[X_S]} - k_R E_t[X_I]$$

$$\mathbb{E}[X_S^2 X_I] =? \; 2E_t[X_S]\,E[X_S X_I] + E[X_S^2]\,E[X_I] - 2\mathbb{E}[X_S]^2 E[X_I]$$

$$\mathbb{E}\left[(X_S-\mu_S)^2(X_I-\mu_I)\right] = 0$$

$$\mathbb{E}\left[(X_S^2 - 2\mu_S X_S + \mu_S^2)(X_I-\mu_I)\right]$$

$$\mathbb{E}\left[X_S^2 X_I - 2\mu_S X_S X_I + \mu_S^2 X_I - \mu_I X_S^2 + 2\mu_S \mu_I X_S - \mu_S^2 \mu_I\right]$$