$\boxed{\text{Appendix 1.4.1.}}$ Linear maps $\mathcal{L}: M_d(\mathbb{C}) \to M_d(\mathbb{C})$

- Define a scalar product on the linear space of $\mathcal{L}$'s by

$$(\mathcal{L}_1, \mathcal{L}_2) = d^2 \operatorname{Tr}\left(\left(\mathcal{L}_1 \otimes id\left[P^{(d)}_{unif}\right]\right)^\dagger \left(\mathcal{L}_2 \otimes id\left[P^{(d)}_{unif}\right]\right)\right)$$

$$= \sum_{a,b;c,f=1}^{d} \operatorname{Tr}\left(\left(\mathcal{L}_1[|a\rangle\langle b|] \otimes |a\rangle\langle b|\right)^\dagger \left(\mathcal{L}_2[|c\rangle\langle f|] \otimes |c\rangle\langle f|\right)\right)$$

$$= \sum_{a,b;f=1}^{d} \operatorname{Tr}\left(\left(\mathcal{L}_1[|a\rangle\langle b|]\right)^\dagger \mathcal{L}_2([|a\rangle\langle f|]) \otimes |b\rangle\langle f|\right)$$

$$= \sum_{a,b;f=1}^{d} \operatorname{Tr}\left(\left(\mathcal{L}_1[|a\rangle\langle b|]\right)^\dagger \mathcal{L}_2([|a\rangle\langle f|])\right) \operatorname{Tr}\left(|b\rangle\langle f|\right)$$

$$= \sum_{a,b=1}^{d} \operatorname{Tr}\left(\left(\mathcal{L}_1[|a\rangle\langle b|]\right)^\dagger \mathcal{L}_2([|a\rangle\langle b|])\right)$$

- Let $\mathcal{F}_{\alpha\beta}: M_d(\mathbb{C}) \to M_d(\mathbb{C})$; $\mathcal{F}_{ab}[X] = F_a^\dagger X F_b \quad \forall X \in M_d(\mathbb{C})$

where $\operatorname{Tr}(F_\alpha^\dagger F_\beta) = \delta_{\alpha\beta} \quad \alpha,\beta = 1, \dots, d^2$ (ONB in $M_d(\mathbb{C})$)

$$(\mathcal{F}_{\alpha\beta}, \mathcal{F}_{\gamma\delta}) = \sum_{a,b=1}^{d} \operatorname{Tr}\left(F_\beta^\dagger |b\rangle\langle a| F_\alpha F_\gamma^\dagger |a\rangle\langle b| F_\delta\right) = \operatorname{Tr}\left(F_\delta F_\beta^\dagger\right) \operatorname{Tr}\left(F_\alpha F_\gamma^\dagger\right) = \delta_{\alpha\gamma}\delta_{\beta\delta}$$

- $\mathcal{F}_{\alpha\beta}$ form an ONB in the space of linear maps and

$$\mathcal{L} = \sum_{\alpha,\beta=1}^{d^2} \mathcal{L}_{\alpha\beta}\, \mathcal{F}_{\alpha\beta} \quad \forall \quad \mathcal{L}: M_d(\mathbb{C}) \to M_d(\mathbb{C})$$

where $\mathcal{L}_{\alpha\beta} = (\mathcal{F}_{\alpha\beta}, \mathcal{L}) = d^2\, \mathrm{Tr}\left(\left(\mathcal{F}_{\alpha\beta} \otimes id_d[P^{(d)}_{mix}]\right)^\dagger \mathcal{L} \otimes id_d\,[P^{(d)}_{mix}]\right)$

$$\boxed{\mathcal{L}[X] = \sum_{\alpha,\beta=1}^{d^2} \mathcal{L}_{\alpha\beta}\; F_\alpha^\dagger X F_\beta}$$

$\boxed{\text{Proposition 1.4.2}}$ $\mathcal{L}: M_d(\mathbb{C}) \to M_d(\mathbb{C})$ is Completely Positive iff $[\mathcal{L}_{\alpha\beta}] \geq 0$.

Proof: 1) $\mathcal{L}$ CP $\Rightarrow$ $\mathcal{L}[X] = \sum_a L_a^\dagger X L_a$

write $L_a = \sum_{\alpha=1}^{d^2} d_{a\alpha} F_\alpha$. Then

$$\mathcal{L}[X] = \sum_a \sum_{\alpha,\beta} d_{a\alpha}\, d_{a\beta}^*\; F_\beta^\dagger X F_\alpha \Rightarrow \mathcal{L}_{\beta\alpha} = \sum_a d_{a\beta}^* d_{a\alpha}$$

$$\forall\ |\psi\rangle \in \mathbb{C}^{d^2}: \langle\psi|[\mathcal{L}_{\alpha\beta}]|\psi\rangle = \sum_a \sum_{\alpha,\beta} \psi_\alpha^* d_{a\alpha}^* \psi_\beta d_{a\beta} = \sum_a \left|\sum_\beta d_{a\beta}\, \psi_\beta\right|^2 \geq 0$$

(

2) $\Lambda = [\ell_{\alpha\beta}] \geq 0 \implies \Lambda = U^\dagger D U \; ; \; \ell_{\alpha\beta} = U^\dagger_{\mu\alpha} U_{\mu\beta} \ell_\mu \, , \; \ell_\mu \geq 0$

$D = \mathrm{diag}(\ell_1, \ell_2, \dots, \ell_{d^2})$

$$\mathcal{L}[X] = \sum_{\alpha,\beta=1}^{d^2} \ell_{\alpha\beta} F_\alpha^\dagger X F_\beta = \sum_{\mu=1}^{d^2} \ell_\mu \sum_{\alpha,\beta=1}^{d^2} U^*_{\mu\alpha} U_{\mu\beta} F_\alpha^\dagger X F_\beta$$

$$= \sum_{\mu=1}^{d^2} \left(\sqrt{\ell_\mu} \sum_{\alpha=1}^{d^2} U^*_{\mu\alpha} F_\alpha^\dagger \right) X \left(\sqrt{\ell_\mu} \sum_{\beta=1}^{d^2} U_{\mu\beta} F_\beta \right)$$

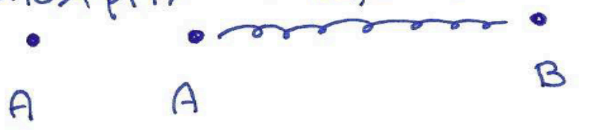
$$= \sum_{\mu=1}^{d^2} L_\mu^\dagger X L_\mu \; ; \; \boxed{L_\mu := \sqrt{\ell_\mu} \sum_{\beta=1}^{d^2} U_{\mu\beta} F_\beta}$$

$\boxed{\text{Exercise 1.4.5}}$ Show that $\mathcal{L}[X] = \sum_k \ell_k L_k^\dagger X L_k \;\; \forall X \in M_d(\mathbb{C})$

defines a CP map iff $\ell_k \geq 0 \;\; \forall k$.

Appendix 1.4.2 Teleportation

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \qquad |\psi^{(2)}_{unif}\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$$



$$|\psi\rangle \otimes |\psi^{(2)}_{unif}\rangle = \frac{1}{\sqrt{2}}\Big(\alpha\big(|000\rangle + |011\rangle\big) + \beta\big(|100\rangle + |111\rangle\big)\Big)$$

$$\Big\downarrow\; U^{A}_{CNOT} \otimes 1_B$$

$$U^{A}_{CNOT} \otimes 1_B\big(|\psi\rangle \otimes |\psi^{(2)}_{unif}\rangle\big) = \frac{1}{\sqrt{2}}\Big(\alpha\big(|000\rangle + |011\rangle\big) + \beta\big(|110\rangle + |101\rangle\big)\Big)$$

$$\Big\downarrow\; (H^{A} \otimes 1_A) \otimes 1_B$$

$$\Big((H^{A} \otimes 1_A)U^{A}_{CNOT}\Big) \otimes 1_B\big(|\psi\rangle \otimes |\psi^{(2)}_{unif}\rangle\big) = \frac{1}{2}\Big(\alpha\Big(\big(|000\rangle + |100\rangle\big) + \big(|011\rangle + |111\rangle\big)\Big) +$$

$$\beta\Big(\big(|010\rangle - |110\rangle\big) + \big(|001\rangle - |101\rangle\big)\Big)\Big) =$$

$$= \frac{1}{2}\Big(|00\rangle \otimes (\alpha|0\rangle + \beta|1\rangle) + |01\rangle \otimes (\alpha|1\rangle + \beta|0\rangle) + |10\rangle \otimes (\alpha|0\rangle - \beta|1\rangle) + |11\rangle \otimes (\alpha|1\rangle - \beta|0\rangle)\Big)$$

$$|\psi\rangle \otimes |\psi^{(2)}_{unif}\rangle \longrightarrow \frac{1}{2}\Big(|00\rangle \otimes |\psi\rangle + |01\rangle \otimes \sigma_1 |\psi\rangle + |10\rangle \otimes \sigma_3 |\psi\rangle + |11\rangle \otimes (-i\sigma_2)|\psi\rangle \Big)$$

Alice measures the orthonormal projections $|ij\rangle\langle ij|$; $i,j = 0,1$

Result $|00\rangle$ communicated to Bob: its state is $|\psi\rangle$
Result $|01\rangle$ communicated to Bob: its state is $\sigma_1|\psi\rangle$
Result $|10\rangle$ communicated to Bob: its state is $\sigma_3|\psi\rangle$
Result $|11\rangle$ communicated to Bob: its state is $-i\sigma_2|\psi\rangle$

Before communication Bob's state is $\frac{1}{2}$: no superluminal communication

After communication Alice's qubit state is different from $|\psi\rangle$: No-cloning

$\boxed{\text{Appendix 1.4.3}}$ No-cloning Theorem (No quantum Xerox machine)

Quantum copying: $\underbrace{|\psi\rangle \otimes |\psi_0\rangle}_{\text{input state}} \xrightarrow{U_{copy}} \underbrace{|\psi\rangle \otimes |\psi\rangle}_{\text{output}}$ $\quad |\psi_0\rangle$: ready state $\quad U_{copy}$ unitary operator on $\mathcal{H}_S \otimes \mathcal{H}_{copy}$

$\boxed{\text{Lemma 1.4.4.}}$ U_{copy} cannot exist

Proof:

$$U_{copy} |\psi_1 \otimes \psi_0\rangle = |\psi_1 \otimes \psi_1\rangle$$

$$U_{copy} |\psi_2 \otimes \psi_0\rangle = |\psi_2 \otimes \psi_2\rangle$$

$$\langle \psi_1 \otimes \psi_1 | \psi_2 \otimes \psi_2 \rangle = \left(\langle \psi_1 | \psi_2 \rangle\right)^2 = \langle \psi_1 \otimes \psi_0 | U^\dagger_{copy} U_{copy} | \psi_2 \otimes \psi_0 \rangle$$

$$= \langle \psi_1 \otimes \psi_0 | \psi_2 \otimes \psi_0 \rangle = \langle \psi_1 | \psi_2 \rangle$$

Then $\langle \psi_1 | \psi_2 \rangle = \begin{cases} 0 & |\psi_1\rangle \text{ and } |\psi_2\rangle \text{ must be orthogonal} \\ 1 & |\psi_1\rangle \text{ and } |\psi_2\rangle \text{ are the same state} \end{cases}$

$\boxed{\text{Appendix 1.4.4.}}$ POVMs

$$\mathcal{L}_X[\rho] = \sum_{i \in I} X_i \rho X_i^\dagger \quad , \quad \sum_{i \in I} X_i^\dagger X_i = \mathbb{1}$$

- $\mathcal{L}_X[|\psi\rangle\langle\psi|] = \sum_{i \in I} X_i |\psi\rangle\langle\psi| X_i^\dagger$

$\sum_{i \in J \subseteq I} \langle\psi| X_i^\dagger X_i |\psi\rangle = \sum_{i \in J \subseteq I} \| X_i|\psi\rangle \|^2$: Probability of measuring the POVM index $i \in I$ in the subset $J \subseteq I$.

$\boxed{\text{Question}}$: can one perfectly discriminate between non orthogonal state vectors $|\psi_1\rangle, |\psi_2\rangle$ such that $\langle\psi_1|\psi_2\rangle \neq 0$?

$\boxed{\text{NO}}$

Proof: if so there must exist a POVM $\{X_i\}_{i\in I = I_1 \cup I_2}$

such that

$$\sum_{i\in I_1} \langle \psi_1 | X_i^\dagger X_i \psi_1 \rangle = 1 \quad \text{and} \quad \sum_{i\in I_1} \langle \psi_2 | X_i^\dagger X_i \psi_2 \rangle$$

$$\overset{\circledast}{} \quad X_i|\psi_2\rangle = 0 \;\; \forall i \in I_1 \;\Longleftarrow\; \sum_{i\in I_1} \| X_i |\psi_2\rangle \|^2 = 0$$

$$\sum_{i\in I_2} \langle \psi_1 | X_i^\dagger X_i \psi_1 \rangle = 0 \quad \text{and} \quad \sum_{i\in I_2} \langle \psi_2 | X_i^\dagger X_i \psi_2 \rangle = 1$$

$$\sum_{i\in I_2} \| X_i |\psi_1\rangle \|^2 = 0 \;\Longrightarrow\; X_i|\psi_1\rangle = 0 \;\; \forall i \in I_2 \quad \circledast\circledast$$

Then,
$$\langle \psi_1 | \psi_2 \rangle = \sum_{i\in I} \langle \psi_1 | X_i^\dagger X_i | \psi_2 \rangle$$
$$= \sum_{i\in I_1} \langle \psi_1 | X_i^\dagger \underbrace{X_i|\psi_2\rangle}_{=0} + \sum_{i\in I_2} \underbrace{\langle \psi_1 | X_i^\dagger}_{=0} X_i |\psi_2\rangle = 0$$
$$\circledast\circledast \qquad\qquad\qquad \circledast$$