TEORIE SUPERSIMMETRICHE in $d=1$

(SUSY QM)

$$L = \frac{\dot{x}^2}{2} \underbrace{- \frac{1}{2}\left(h'(x)\right)^2}_{-V(x)} + \underbrace{\frac{i}{2}\left(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi\right) - h''(x)\bar{\psi}\psi}_{\text{completamento supersimmetrico di variazione bosonica}}$$

$-V(x)$ ora ha una forma specifica dettata dal richiere inv. per susy

- $\psi, \bar{\psi}$ sono variabili anticommutanti (a valori in Grassmann)
- $\bar{\psi} = \psi^\dagger$ (c.c.) $\qquad \psi = \theta^1 + i\theta^2$
- $L \in \mathbb{R}$ $\qquad \bar{\psi} = \theta^1 - i\theta^2$

$$\psi^2 = (\theta^1 + i\theta^2)(\theta^1 + i\theta^2) = \cancel{(\theta^1)^2} + i\theta^2\theta^1 + i\theta^1\theta^2 - \cancel{(\theta^2)^2} = 0$$

$$(\bar{\psi}\psi)^\dagger = \psi^\dagger \bar{\psi}^\dagger = \bar{\psi}\psi \leftarrow \mathbb{R}$$

$$(\bar{\psi}\dot{\psi})^\dagger = \dot{\bar{\psi}}\psi \leftarrow \bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi \in i\mathbb{R}$$

Trasf. di susy: $\leftarrow$ simmetria globale (susy rigida) ϵ non dip. dalle coord. dello spazio-tempo M.

$$\delta x = \epsilon\bar{\psi} - \bar{\epsilon}\psi$$

$$\delta\psi = \epsilon\left(i\dot{x} + h'(x)\right)$$

$$\delta\bar{\psi} = \bar{\epsilon}\left(-i\dot{x} + h'(x)\right)$$

$\epsilon = \epsilon_1 + i\epsilon_2$ — ϵ_1, ϵ_2 numeri di Grassmann

$\downarrow$

$$\delta L = \frac{d}{dt}(\dots) \Rightarrow \text{Azione è invariante}$$

(if bndry variations vanish)

$$\delta S = \int_0^\beta \delta L \, dt = 0$$

Osservazione:

$$[\delta_1, \delta_2] x = 2i(\epsilon_1\bar{\epsilon}_2 - \epsilon_2\bar{\epsilon}_1)\dot{x}$$

$$[\delta_1, \delta_2] \psi = 2i(\epsilon_1\bar{\epsilon}_2 - \epsilon_2\bar{\epsilon}_1)\dot{\psi}$$

Teorema di Nöther : $\delta L = 0 \Rightarrow P = \sum_h \frac{\partial L}{\partial \dot{q}_h}\frac{\partial \delta q_h}{\partial \alpha}$ cost. del moto

ma se $\underset{\overset{|||}{\frac{d}{dt}P}}{\frac{\partial L}{\partial \alpha}} = \frac{d}{dt}\Lambda \Rightarrow \frac{d}{dt}(\underbrace{P-\Lambda}) = 0$

cost. del moto quando δL è una derivata totale

Calcoliamo Λ :

$$L = \frac{\dot{x}^2}{2} - \frac{1}{2}(h'(x))^2 + \frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - h''(x)\bar{\psi}\psi$$

$$\delta L = \dot{x}\delta\dot{x} - h'h''\delta x + \frac{i}{2}(\delta\bar{\psi}\dot{\psi} + \bar{\psi}\delta\dot{\psi} - \delta\dot{\bar{\psi}}\psi - \dot{\bar{\psi}}\delta\psi) - \underbrace{\cancel{h'''\delta x \bar{\psi}\psi}}_{=0} - h''\delta\bar{\psi}\psi - h''\bar{\psi}\delta\psi$$

$=0$ δx omogeneo in $\psi, \bar{\psi}$

$\delta x = \epsilon\bar{\psi} - \bar{\epsilon}\psi$
$\delta\psi = \epsilon(i\dot{x} + h'(x))$
$\delta\bar{\psi} = \bar{\epsilon}(-i\dot{x} + h'(x))$

$$= \dot{x}(\epsilon\dot{\bar{\psi}} - \bar{\epsilon}\dot{\psi}) - h'h''(\epsilon\bar{\psi} - \bar{\epsilon}\psi) + \frac{i}{2}\{\bar{\epsilon}(-i\dot{x}+h')\dot{\psi} + \bar{\psi}\epsilon(i\ddot{x}+h''\dot{x})$$

$$- \bar{\epsilon}(-i\ddot{x}+h''\dot{x})\psi - \dot{\bar{\psi}}\epsilon(i\dot{x}+h')\} - h''(-i\dot{x}+h')\bar{\epsilon}\psi - h''\bar{\psi}\epsilon(i\dot{x}+h')$$

$$= \epsilon\{\underline{\dot{x}\dot{\bar{\psi}}} - \cancel{h'h''\bar{\psi}} - \frac{i}{2}\bar{\psi}(i\ddot{x} + \underline{\underline{h''\dot{x}}}) + \frac{i}{2}\dot{\bar{\psi}}(\underline{i\dot{x}} + h') + h''\bar{\psi}(\underline{\underline{i\dot{x}}} + \cancel{h'})\}$$

$+$ c.c. (quando si prende c.c., cioè $\dagger$, variabili di Grassmann si scambiano ordine)

$$= \frac{\epsilon}{2}\{\underbrace{\dot{x}\dot{\bar{\psi}} + \ddot{x}\bar{\psi}} + \underbrace{ih''\dot{x}\bar{\psi} + ih'\dot{\bar{\psi}}}\} + \text{c.c.}$$

$$= \frac{\epsilon}{2} \frac{d}{dt}\left[\dot{x}\bar{\psi} + i h'\bar{\psi}\right]^{\text{ecc.}} \Rightarrow \quad \Lambda_\epsilon = \frac{1}{2}\dot{x}\bar{\psi} + \frac{i}{2}h'\bar{\psi}$$

$$\Lambda_{\bar{\epsilon}} = \Lambda_\epsilon^\dagger$$

$$P = \sum_h \frac{\partial L}{\partial \dot{q}_h}\frac{\partial \delta q_h}{\partial \alpha}$$

$$\frac{\partial L}{\partial \dot{x}} = \dot{x} \qquad \frac{\partial}{\partial \epsilon}\delta x = \bar{\psi} \qquad \frac{\partial}{\partial \bar{\epsilon}}\delta x = -\psi$$

$$\frac{\partial L}{\partial \dot{\psi}} = -\frac{i}{2}\bar{\psi} \qquad \frac{\partial}{\partial \epsilon}\delta\psi = i\dot{x} + h' \qquad \frac{\partial}{\partial \bar{\epsilon}}\delta\psi = 0$$

$$\frac{\partial L}{\partial \dot{\bar{\psi}}} = -\frac{i}{2}\psi \qquad \frac{\partial}{\partial \epsilon}\delta\bar{\psi} = 0 \qquad \frac{\partial}{\partial \bar{\epsilon}}\delta\bar{\psi} = -i\dot{x} + h'$$

$$P_\epsilon = \dot{x}\bar{\psi} + \frac{1}{2}\dot{x}\bar{\psi} - \frac{i}{2}h'\bar{\psi} = \frac{3}{2}\dot{x}\bar{\psi} - \frac{i}{2}h'\bar{\psi}$$

$$P_{\bar{\epsilon}} = -\dot{x}\psi - \frac{1}{2}\dot{x}\psi - \frac{i}{2}h'\psi = -\frac{3}{2}\dot{x}\psi - \frac{i}{2}h'\psi$$

Def. le quantità conservate:

$$\boxed{\begin{aligned} Q &= i\left(P_\epsilon - \Lambda_\epsilon\right) = i\dot{x}\bar{\psi} + h'\bar{\psi} \\ \bar{Q} &= i\left(P_{\bar{\epsilon}} - \Lambda_{\bar{\epsilon}}\right) = -i\dot{x}\psi + h'\psi \end{aligned}} \qquad Q^\dagger = \bar{Q}$$

$\hookrightarrow$ <u>CARICHE di SUPERSIMMETRIA</u>

Quantizzazione canonica

Riscriviamo la Lagrangiana (a meno di una derivata totale)

$$L = \frac{\dot{x}^2}{2} - \frac{1}{2}(h')^2 + \frac{i}{2}(\bar{\psi}\dot{\psi} - \underbrace{\dot{\bar{\psi}}\psi}_{-\frac{d}{dt}(\bar{\psi}\psi) + \bar{\psi}\dot{\psi}}) - h''\bar{\psi}\psi$$

$$= \frac{\dot{x}^2}{2} - \frac{1}{2}(h')^2 + i\underbrace{\bar{\psi}\dot{\psi}}_{\substack{\text{Lagrangiana di Dirac} \\ \text{per un fermione massless}}} - h''\bar{\psi}\psi$$

Momenti coniugati:

$$p = \frac{\partial L}{\partial \dot{x}} = \dot{x}$$

$$\Pi = \frac{\partial L}{\partial \dot{\psi}} = i\bar{\psi}$$

$$\bar{\Pi} = \frac{\partial L}{\partial \dot{\bar{\psi}}} = 0$$

$$H = p\dot{x} + \Pi\dot{\psi} + \bar{\Pi}\dot{\bar{\psi}} - L$$

$$= \frac{p^2}{2} + \frac{1}{2}(h')^2 + \underbrace{\frac{1}{2}h''(\bar{\psi}\psi - \psi\bar{\psi})}_{\substack{\text{abbiamo scelto di scrivere} \\ \text{campi in pto ordine}}}$$

Regole di quantizzazione canoniche

$$\begin{cases} [x, p] = i \\ \{\psi, \bar{\psi}\} = 1 \end{cases} \qquad (*)$$

Spazio di Hilbert (rappresentaz. dell'algebra $(*)$)

$$\hat{x}\,\Psi(x) = x\,\Psi(x) \qquad \Psi(x) \in L^2(\mathbb{R})$$

$$\hat{p}\,\Psi(x) = -i\frac{d}{dx}\Psi(x)$$

Per $\psi, \bar{\psi}$: $\quad \{\psi,\psi\} = 0 \quad \{\bar{\psi},\bar{\psi}\} = 0 \quad \{\psi,\bar{\psi}\} = 1$

analogo a $\quad [a, a] = 0 \quad [a^+, a^+] = 0 \quad [a, a^+] = 1$

Def. l'op. FERMION NUMBER

$$F = \bar{\psi}\psi$$

e vediamo che F è t.c.

$$[F, \psi] = -\psi$$

$$[F, \bar\psi] = \bar\psi$$

$[F,\psi] = \bar\psi\cancel{\psi}\cdot\psi - \psi\bar\psi\psi =$

$= -\psi(\bar\psi\psi + \underline{\psi\bar\psi}) = -\psi$

$\bar\psi|\phi\rangle$ t.c. $F|\phi\rangle = f|\phi\rangle \Rightarrow F(\bar\psi|\phi\rangle) = (\bar\psi F + \bar\psi)|\phi\rangle = (f+1)\bar\psi|\phi\rangle$

Definiamo uno stato $|0\rangle$ t.c. $\psi|0\rangle = 0$, allora

$|1\rangle = \bar\psi|0\rangle$. Ma $|2\rangle \overset{?}{=} \bar\psi|1\rangle = \underbrace{\bar\psi\bar\psi}_{=0}|0\rangle = 0$.

$\Rightarrow$ spazio con solo due vett. indip.

$$V_F = \langle\, |0\rangle, \bar\psi|0\rangle \,\rangle$$

↳ in questa base possiamo rappr. ψ e $\bar\psi$ con le matrici

$$\psi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \bar\psi \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\mathcal{H} = L^2(\mathbb{R}) \otimes V_F$$