Solution problem set 5

Ex 1) Find all mixed strategy Nash equilibrium of the following game (you have to use the property of the Nash equilibrium in mixed strategies)

		Player 2		
		L	M	R
Player 1	T	2, 2	0, 3	1, 3
	B	3, 2	1, 1	0, 2

Notation

t is the probability to play T

l is the probability to play L

m is the probability to play M

Player 1's strategy *(t, 1-t)*

Player 2's strategy *(l, m, 1 – l - m)*

E(X) expected value from playing the pure strategy X

		Player 2		
		L	M	R
Player 1	T	2, 2	0, 3	1, 3
	B	3, 2	1, 1	0, 2

We start considering all possible cases for player 1:

$s_1 = (1, 0)$,

$s_1 = (0, 1)$,

$s_1 = (t, 1 - t)$ for $0 < t < 1$

For each one we explore if it can be played in a NE

		Player 2		
		L	M	R
Player 1	T	2, 2	0, 3	1, 3
	B	3, 2	1, 1	0, 2

Case $s_1 = (1, 0)$

it must be: $E(T) \geq E(B)$

Now we look at the expected payoff of player 2 when $s_1 = (1, 0)$

$E(M) = E(R) = 3 > E(L) = 2$

Then a mixed strategy for player 2 must be of the type $(0, m, 1 - m)$

Given this possible player 2's strategy we compute the expected payoff of player 1

$E(T) = 1 - m, E(B) = m$

In equilibrium $s_1 = (1, 0)$ only if $E(T) \geq E(B)$ is satisfied. This is true only if $m \leq 0.5$

Therefore all strategy profiles $(1, 0)$ $(0, m, 1 - m)$ with $m \leq 0.5$ are Nash equilibria

Note

1. there are infinite equilibria (one for each possible value of $m \leq 0.5$)
2. This set of equilibria includes the NE in pure strategy $\{(1, 0)\ (0, 0, 1)\}$

    (Player 1 plays T and Player 2 plays R)

		Player 2		
		L	M	R
Player 1	T	2, 2	0, 3	1, 3
	B	3, 2	1, 1	0, 2

Case $\boldsymbol{s_1 = (0, 1)}$

It must be: E(B) ≥ E(T)

Now we look at the expected payoff of player 2 when $\boldsymbol{s_1 = (0, 1)}$

$E(L) = E(R) = 2 > E(M) = 1$

Then a mixed strategy for player 2 must be of the type $(l, 0, 1 - l)$

Given this possible player 2's strategy we compute the expected payoff of player 1

$E(T) = 1 + l, E(B) = 3l$

In equilibrium $s_1 = (0, 1)$ only if $E(B) \geq E(T)$ is satisfied. This is true only if $l \geq 0.5$

Therefore all strategy profiles $(0, 1)$ $(l, 0, 1 - l)$ with $l \geq 0.5$ are Nash equilibria

Note

1. there are infinite equilibria
2. This set of equilibria includes the NE in pure strategy $\{(0, 1)\ (1, 0, 0)\}$

   (Player 1 plays B and Player 2 plays L)

Case $s_1 = (t, 1 - t)$ for $0 < t < 1$

		Player 2		
		L	M	R
Player 1	T	2, 2	0, 3	1, 3
	B	3, 2	1, 1	0, 2

It must be: $E(B) = E(T)$

Now we look at the expected payoff of player 2 when $s_1 = (t, 1 - t)$ **for** $0 < t < 1$

$E(L) = 2$

$E(M) = 3t + (1 - t) = 1 + 2t$

$E(R) = 3t + 2(1 - t) = 2 + t$

We have to explore all the possible classes of player 2's strategy, i.e. for each case we have to verify the equilibrium conditions given that player 1 strategy is of the type $(t, 1 - t)$ for $t \in (0, 1)$.

All these cases are:

- $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$
- $(l, 1 - l, 0)$, $(l, 0, 1 - l)$, $(0, m, 1 - m)$
- $(l, m\ 1 - l - m)$

		Player 2		
		L	M	R
Player 1	T	2, 2	0, 3	1, 3
	B	3, 2	1, 1	0, 2

Player 2's expected payoffs

$E(L) = 2$

$E(M) = 1 + 2t$

$E(R) = 2 + t$

Note that we can reduce the number of cases. For every $t \in (0, 1)$:

- the expected payoff from L is smaller than the expected payoff from R, (E$(L) < E(R)$) ,

  indeed $2 + t > 2$ for $t > 0$.

- the expected payoff from M is smaller than the expected payoff from R, (E$(M) < E(R)$) ,

  indeed 2 + t > 1 + 2 for $t < 1$

Then we can eliminate all cases where either L or M or both are played by strictly positive probability.

Then it remains to explore only strategy (0, 0, 1).

In this case the player 1's expected payoffs are

$E(T) = 1 > E(B) = 0$, Then condition $E(T) = E(B)$ does not hold

Final results:

There are two sets of Nash equilibria:

- (1, 0) (0, m, 1 –m) with m ≤ 0.5
- (0, 1) (l, 0, 1 – l) with l ≥ 0.5

Note that each set contains an equilibrium in pure strategies, respectively, (1, 0) (0, 0, 1) and (0, 1) (1, 0, 0).

Ex 2)

Each of two firms has a job opening. The firms offer different wages: firm i offers wage w_i where $0.5 \cdot w_1 < w_2 < 2 \cdot w_1$.

There are two workers that want to apply for a job.

Each of whom can apply to only one firm. The workers simultaneously decide whether apply to firm 1 or to firm 2.

If only one worker applies to a given firm, that worker gets the job. If both workers apply to one firm, the firm hires one worker at random and the other worker remains unemployed.

1) Rapresent this game using the normal form
2) Solve for the Nash equilibria (pure and mixed strategies)

Normal form

		Worker 2	
		Firm 1	Firm 2
Worker 1	Firm 1	$0.5 \cdot w_1, 0.5 \cdot w_1$	w_1, w_2
	Firm 2	w_2, w_1	$0.5 \cdot w_2, 0.5 \cdot w_2$

$$0.5 \cdot w_1 < w_2 < 2 \cdot w_1.$$

		Worker 2	
		Firm 1	Firm 2
Worker 1	Firm 1	$0.5 \cdot w_1, 0.5 \cdot w_1$	w_1, w_2
	Firm 2	w_2, w_1	$0.5 \cdot w_2, 0.5 \cdot w_2$

Notation

- p is the probability that Player 1 plays Firm 1
- q is the probability that Player 2 plays Firm 1
- the player 1's mixed strategy is $s_1 = (p, 1 - p)$
- the player 2's mixed strategy is $s_2 = (q, 1 - q)$

We compute the expected values for each single action given a strategy of the opponent

Player 1

$$E_1(Firm1|s_2) = q\frac{w_1}{2} + (1-q)w_1 = w_1(1 - \frac{q}{2})$$
$$E_1(Firm2|s_2) = qw_2 + (1-q)\frac{w_2}{2} = w_2(\frac{1+q}{2})$$

Player 2

$$E_2(Firm1|s_1) = p\frac{w_1}{2} + (1-p)w_1 = w_1(1 - \frac{p}{2})$$
$$E_2(Firm2|s_1) = pw_2 + (1-p)\frac{w_2}{2} = w_2(\frac{1+p}{2})$$

Suppose s_1 = (1, 0) , $p = 1$

$E_1(Firm1|s2) = w_1(1 - \frac{q}{2})$

$E_1(Firm2|s2) = w_2(\frac{1+q}{2})$

$E_2(Firm1|s1) = w_1(1 - \frac{p}{2})$

$E_2(Firm2|s1) = w_2(\frac{1+p}{2})$

Player 2’ s expected profits are:

$E_2(Firm1|s_1) = \frac{w1}{2}$

$E_2(Firm2|s_1) = w_2$

Given that $w_2 > \frac{w1}{2}$ the best response for player 2 is $s_2 = (0, 1)$

Given $s_2 = (0, 1)$ (then $q = 0$) the expected payoff of player 1 are:

$E_1(Firm1|s_2) = w_1$

$E_1(Firm2|s_2) = \frac{w_2}{2}$

Given that $w_1 > \frac{w_2}{2}$ the best response for player 1 is $s_1 = (1, 0)$

Then $s_1 = (1, 0)$ $s_2 = (0, 1)$ is a Nash equilibrium

Suppose $s_1 = (0, 1\ )\ p = 0$

$$E_1(Firm1|s2) = w_1(1 - \frac{q}{2})$$
$$E_1(Firm2|s2) = w_2(\frac{1+q}{2})$$
$$E_2(Firm1|s1) = w_1(1 - \frac{p}{2})$$
$$E_2(Firm2|s1) = w_2(\frac{1+p}{2})$$

Player 2' s expected profits are:

$$E_2(Firm1|s_1) = w_1$$
$$E_2(Firm2|s_1) = 0.5w_2$$

Given that $w_1 > \frac{w2}{2}$ the best response for player 2 is $s_2 = (1, 0)$

Given $s_2 = (1, 0\ )\ (q = 1)$ the expected payoff of player 1 are:

$$E_1(Firm1|s_2) = 0.5w_1$$
$$E_1(Firm2|s_2) = w_2$$

Given that $w_2 > \frac{w_1}{2}$ the best response for player 1 is $s_1 = (0, 1)$

Then $s_1 = (0, 1\ )\ s_2 = (1, 0\ )$ is a Nash equilibrium

Suppose $s_1 = (p, 1 - p)$ for $p \in (0,1)$

$E_1(Firm1|s2) = w_1(1 - \frac{q}{2})$

$E_1(Firm2|s2) = w_2(\frac{1+q}{2})$

$E_2(Firm1|s1) = w_1(1 - \frac{p}{2})$

$E_2(Firm2|s1) = w_2(\frac{1+p}{2})$

In this case must be that $E_1(Firm1|s_2) = E_1(Firm2|s_2)$ i.e.

$$w_1\left(1 - \frac{q}{2}\right) = w_2\left(\frac{1+q}{2}\right)$$

This is true only if $q = \frac{2w_1 - w_2}{w_1 + w_2}$, i.e. $s_2 = \left(\frac{2w_1 - w_2}{w_1 + w_2}, \frac{2w_2 - w_1}{w_1 + w_2}\right)$

But for player 2, in order to play such a strategy, the condition

$E_2(Firm1|s_1) = E_2(Firm2|s_1)$

must be satisfied, i.e.:

$$w_1\left(1 - \frac{p}{2}\right) = w_2\left(\frac{1+p}{2}\right)$$

This is true only if $p = \frac{2w_1 - w_2}{w_1 + w_2}$, i.e. $s_1 = \left(\frac{2w_1 - w_2}{w_1 + w_2}, \frac{2w_2 - w_1}{w_1 + w_2}\right)$

Then the strategy profile $s_1 = \left(\frac{2w_1 - w_2}{w_1 + w_2}, \frac{2w_2 - w_1}{w_1 + w_2}\right)$ $s_2 = \left(\frac{2w_1 - w_2}{w_1 + w_2}, \frac{2w_2 - w_1}{w_1 + w_2}\right)$ is a Nash equilibrium.