EQ. DI HAMILTON

Sistemi Hamiltoniani: sistemi la cui dinamica è determinata da eq. differenziali nella forma delle eq. di Hamilton

Data funz. Hamiltoniana $H(\bar{p}, \bar{q}, t)$ def. sullo SPAZIO DELLE FASI, le eq. del moto sono

$$\begin{cases} \dot{p}_\ell = -\dfrac{\partial H}{\partial q_\ell} \\ \dot{q}_\ell = \dfrac{\partial H}{\partial p_\ell} \end{cases} \qquad \ell = 1, \dots, n$$

→ Risolte le equazioni, trovo una traiettoria nello spazio delle fasi, data $p_h(t)$ $q_h(t)$ $h = 1, \dots, n$

Se $\det \dfrac{\partial^2 H}{\partial p_h \partial p_k} \neq 0 \Rightarrow$ posso invertire la seconda equazione e ricavare $p_\ell = p_\ell(\bar{q}, \dot{\bar{q}}, t)$

Trasformata di Legendre

$$L(\bar{q}, \dot{\bar{q}}, t) \longleftrightarrow H(\bar{p}, \bar{q}, t) = \bar{p} \cdot \dot{\bar{q}} - L(\bar{q}, \dot{\bar{q}}, t)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_h} - \frac{\partial L}{\partial q_h} = 0 \longleftrightarrow \begin{cases} \dot{p}_h = -\dfrac{\partial H}{\partial q_h} \\ \dot{q}_h = \dfrac{\partial H}{\partial p_h} \end{cases}$$

n eq. diff. 2° ord. $\qquad$ $2n$ eq. diff. 1° ord.

Esempi:

1) PTO MATERIALE in coord. cartesiane

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \qquad V = V(x, y, z)$$

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z)$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} = p_x(\bar{q}, \dot{\bar{q}}, t) \overset{\text{invers.}}{\rightsquigarrow} \dot{x} = \frac{p_x}{m}$$

$$p_y = \quad = m\dot{y} \qquad \dot{y} = \frac{p_y}{m} \qquad (*)$$

$$p_z = \quad = m\dot{z} \qquad \dot{z} = p_z/m$$

$$H(p_x, p_y, p_z, x, y, z) = p_x \dot{x} + p_y \dot{y} + p_z \dot{z} - L \Big|_{(*)}$$

$$= p_x \left(\frac{p_x}{m}\right) + p_y \left(\frac{p_y}{m}\right) + p_z \left(\frac{p_z}{m}\right) - \frac{m}{2}\left(\left(\frac{p_x}{m}\right)^2 + \left(\frac{p_y}{m}\right)^2 + \left(\frac{p_z}{m}\right)^2\right) + V(x,y,z)$$

$$= \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + V(x,y,z) = \frac{\bar{p}^2}{2m} + V(x,y,z) = T + V$$

2) Pto mat. in coord sferiche

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\varphi}^2\right) - V(r,\theta,\varphi)$$

$$p_r = m\dot{r} \qquad\qquad \dot{r} = p_r/m$$

$$p_\theta = m r^2 \dot{\theta} \quad \overset{\text{inv.}}{\rightsquigarrow} \quad \dot{\theta} = p_\theta / m r^2$$

$$p_\varphi = m r^2 \sin^2\theta\, \dot{\varphi} \qquad\qquad \dot{\varphi} = p_\varphi / m r^2 \sin^2\theta$$

$$\rightsquigarrow \quad H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2\theta}\right) + V(r,\theta,\varphi)$$

3) Oscillatore armonico di freq. ω:

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2 \quad \rightsquigarrow \quad H(p,x) = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$$

4) Forze elettromagnetiche

$$\bar{E} = -\left(\bar{\nabla}\phi + \frac{\partial \bar{A}}{\partial t}\right)$$

$$\bar{F} = e\left(\bar{E} + \underbrace{\dot{\bar{q}} \times \bar{B}}\right) \qquad \bar{B} = \bar{\nabla} \times \bar{A}$$

per $\bar{B}$ cost., la forma è la stessa della forza di Coriolis con $\bar{\omega}$ cost.

$$= -e\left(\bar{\nabla}\phi + \frac{\partial \bar{A}}{\partial t}\right) + e\, \dot{\bar{q}} \times \left(\bar{\nabla} \times \bar{A}\right)$$

$\bar F$ si ricava da un potenziale $V(\bar q, \dot{\bar q}) = \overbrace{e\,\phi(\bar q)}^{V_0} - \overbrace{e\,\dot{\bar q}\cdot \bar A(\bar q)}^{V_1}$

$$\frac{d}{dt}\frac{\partial V}{\partial \dot q_h} = \frac{d}{dt}\left(-e\,A_h(\bar q)\right) = -e\left(\frac{\partial A_h}{\partial t} + \sum_{k=1}^{3} \dot q_k \frac{\partial A_h}{\partial q_k}\right)$$

$$\frac{\partial V}{\partial q_h} = e\,\frac{\partial \phi}{\partial q_h} - e\sum_{k=1}^{3} \dot q_k \frac{\partial A_k}{\partial q_h}$$

$$\frac{d}{dt}\frac{\partial V}{\partial \dot q_h} - \frac{\partial V}{\partial q_h} = -e\left(\frac{\partial \phi}{\partial q_h} + \frac{\partial A_h}{\partial t}\right) + e\sum_{k=1}^{3} \dot q_k \left(\frac{\partial A_k}{\partial q_h} - \frac{\partial A_h}{\partial q_k}\right)$$

$$L = T - V = \frac{1}{2} m\,\dot{\bar q}^2 + e\,\dot{\bar q}\cdot\bar A - e\phi$$

$$p_h = \frac{\partial L}{\partial \dot q_h} = m\,\dot q_h + e\,A_h \Rightarrow \dot q_h = \frac{p_h - e A_h}{m} \qquad (*)$$

$$H = \bar p\cdot\dot{\bar q} - L\Big|_{(*)} = \bar p\cdot\left(\frac{\bar p - e\bar A}{m}\right) - \frac{\cancel{m}}{2}\frac{(\bar p - e\bar A)^2}{m^{\cancel{2}}}$$

$$- e\bar A\cdot\left(\frac{\bar p - e\bar A}{m}\right) + e\phi =$$

$$= \frac{(\bar p - e\bar A)^2}{m} - \frac{(\bar p - e\bar A)^2}{2m} + e\phi = \frac{(\bar p - e\bar A)^2}{2m} + e\phi$$

$$T + V_0$$

FORMULAZIONE VARIAZIONALE delle eq. di Hamilton

Siamo interessati alle traiettorie nello sp. delle fasi

$$\underbrace{p_i(t)\,,\quad q_i(t)}_{\text{sono funzioni}\ :\ \mathbb{R}\to\mathbb{R}^{2n}} \qquad i=1,\dots,n$$

Funzionale AZIONE HAMILTONIANA

$$S[\bar{p},\bar{q}] \equiv \int_{t_0}^{t_1} \underbrace{\left[\sum_{h=1}^{n} p_h(t)\,\dot{q}_h(t) - H(\bar{p}(t),\bar{q}(t),t)\right]}_{=L(\bar{q}(t),\,\dot{\bar{q}}(t),\,t)} dt$$

Se il sist. Ham. è equiv. a un sist. Lagrangiano, S coincide con l'az. Ham. vista in precedenza.

[Se ho un funzionale $S[\bar{p},\bar{q}] = \int dt\; F(\bar{p}(t),\bar{q}(t))$

allora $\delta S = \dfrac{d}{d\alpha}\displaystyle\int dt\; \dfrac{d}{d\alpha}\left[F(\bar{p}+\alpha\delta\bar{p}\,,\;\bar{q}+\alpha\delta\bar{q})\right] =$

$$= \int dt \sum_h \left[\frac{\partial F}{\partial p_h}\delta p_h + \frac{\partial F}{\partial q_h}\delta q_h\right]_{\alpha=1}$$]

In seguito ad arbitrarie variazioni $\delta p_h(t)$ e $\delta q_h(t)$, il funzionale S subisce la variazione

$$\delta S[\bar{p},\bar{q},\delta\bar{p},\delta\bar{q}] = \int_{t_0}^{t_1} \sum_{h=1}^{n}\left[\delta p_h(t)\,\dot{q}_h(t) + p_h(t)\,\delta\dot{q}_h(t) - \frac{\partial H}{\partial p_h}\delta p_h - \frac{\partial H}{\partial q_h}\delta q_h\right]dt$$

$\delta\dot{q}_h = \dfrac{d}{dt}\delta q_h$ e integrando per parti

$$\delta S = \int_{t_0}^{t_1} \sum_{h=1}^{n} \left[\left(\dot q_h - \frac{\partial H}{\partial p_h} \right) \delta p_h - \left(\dot p_h + \frac{\partial H}{\partial q_h} \right) \delta q_h \right] dt + \sum_{h=1}^{n} p_h \, \delta q_h \Big|_{t_0}^{t_1}$$

$$\Downarrow$$

Prop. Il moto $\bar p(t), \bar q(t)$ $(t_0 \leq t \leq t_1)$ rende stazionario il funzionale azione $S[\bar p, \bar q]$ corrispondente a una data Hamiltoniana H, per variazioni arbitrarie e (per $\delta \bar q$) nulle agli estremi se e solo se esso soddisfa le eq. di Hamilton relative ad H.

PARENTESI di POISSON

Cost. del moto : $f(\bar{p}, \bar{q}, t)$ t.c. se la valutiamo sulle soluz. delle eq. di Ham. otteniamo

$$\frac{d}{dt} f(\bar{p}(t), \bar{q}(t), t) = 0 .$$

$\bar{q}(t), \bar{p}(t)$ soddisf. eq. Ham.

$$\frac{d}{dt} f(\bar{p}(t), \bar{q}(t), t) = \frac{\partial f}{\partial t} + \sum_{h=1}^{n} \Big(\frac{\partial f}{\partial p_h} \underbrace{\dot{p}_h}_{= -\frac{\partial H}{\partial q}} + \frac{\partial f}{\partial q_h} \underbrace{\dot{q}_h}_{= \frac{\partial H}{\partial p}} \Big) =$$

$$= \frac{\partial f}{\partial t} + \underbrace{\sum_{h=1}^{n} \Big(\frac{\partial f}{\partial q_h} \frac{\partial H}{\partial p_h} - \frac{\partial f}{\partial p_h} \frac{\partial H}{\partial q_h} \Big)}_{\equiv \{f, H\}}$$

funz. di $(\bar{p}, \bar{q})$ valutata in $\bar{p}(t), \bar{q}(t)$.

PARENTESI DI POISSON

Par. di Poisson tra due funzioni f e g

$$\underbrace{\{f, g\}}_{\text{ancora una funz. di } \bar{p}, \bar{q}} = \sum_{k=1}^{n} \left[\frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} \right]$$

$\{\ ,\ \}$ prende due funzioni def. sullo spazio delle fasi e restituisce un'altra funz. def. sullo sp. delle fasi.

$$\frac{d}{dt} f(\bar{p}(t), \bar{q}(t), t) = \frac{\partial f}{\partial t}(\bar{p}(t), \bar{q}(t), t) + \{f, H\}(\bar{p}(t), \bar{q}(t), t)$$

f è una COSTANTE del MOTO $\iff$ $\frac{\partial f}{\partial t} + \{f, H\} = 0$

(f indip. esp. da t) $\quad$ $(\{f, H\} = 0)$

Formalismo compatto:

Partiamo da un sist. a 1 grado di lib.: spazio delle fasi ha coord. p e q.

Eq. Hamilton:

$$\begin{cases} \dot{p} = -\dfrac{\partial H}{\partial q} \\ \dot{q} = \dfrac{\partial H}{\partial p} \end{cases}$$

Def. $\bar{x} = \begin{pmatrix} p \\ q \end{pmatrix} \rightsquigarrow \nabla_x H = \begin{pmatrix} \dfrac{\partial H}{\partial p} \\ \dfrac{\partial H}{\partial q} \end{pmatrix}$

Consideriamo la matrice $E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Allora le eq. di Hamilton possono essere scritte come

$$\dot{\bar{x}} = \begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} -\partial H/\partial q \\ \partial H/\partial p \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \partial H/\partial p \\ \partial H/\partial q \end{pmatrix} = E\, \bar{\nabla}_x H$$

$$\dot{\bar{x}} = E\, \bar{\nabla}_x H \quad \leftarrow \text{ è della forma } \quad \dot{\bar{x}} = \bar{f}(\bar{x}, t)$$

$$\frac{d}{dt} F(\bar{x}(t), t) = \frac{\partial F}{\partial t} + \underline{\underline{\mathcal{L}_{E\bar{\nabla}H} F}} = \frac{\partial F}{\partial t} + \sum_{i=1}^{2} f_i \frac{\partial F}{\partial x_i} =$$

$$= \frac{\partial F}{\partial t} - \frac{\partial H}{\partial q}\frac{\partial F}{\partial p} + \frac{\partial H}{\partial p}\frac{\partial F}{\partial q} = \frac{\partial F}{\partial t} + \underline{\underline{\{F, H\}}}$$

In generale

$$\{f, g\} = \mathcal{L}_{E\bar{\nabla}g} f = \sum_i \left(E_{ij} \frac{\partial g}{\partial x_j} \right) \frac{\partial f}{\partial x_i} =$$

$$= \bar{\nabla} f \cdot E\, \bar{\nabla} g$$

$$\{g, f\} = \bar\nabla g \cdot \underset{\substack{\uparrow \\ \text{matrice ANTISIMM.}}}{E}\,\bar\nabla f = \sum_{ij} E_{ij} \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial x_j} =$$

$$= -\sum_{ij} E_{ji} \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_i} = -\{f, g\}$$

Prendiamo ora un sistema a n gradi di libertà

$$\bar x = \begin{pmatrix} p_1 \\ \vdots \\ p_n \\ q_1 \\ \vdots \\ q_n \end{pmatrix} = \begin{pmatrix} \bar p \\ \bar q \end{pmatrix} \qquad \bar\nabla_x f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_{2n}} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial p_1} \\ \vdots \\ \frac{\partial f}{\partial p_n} \\ \frac{\partial f}{\partial q_1} \\ \vdots \\ \frac{\partial f}{\partial q_n} \end{pmatrix} = \begin{pmatrix} \bar\nabla_p f \\ \bar\nabla_q f \end{pmatrix}$$

$\downarrow$

$x_1, \dots, x_{2n}$

$$E = \begin{pmatrix} 0 & -\mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix}$$

matrice $2n \times 2n$

Eq. Hamilton: $\dot{\bar x} = E\,\bar\nabla_x H$

$\downarrow$

$$\begin{pmatrix} \dot{\bar p} \\ \dot{\bar q} \end{pmatrix} = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} \bar\nabla_p H \\ \bar\nabla_q H \end{pmatrix} = \begin{pmatrix} -\bar\nabla_q H \\ \bar\nabla_p H \end{pmatrix}$$

cioè in componenti

$$\dot p_h = -\frac{\partial H}{\partial q_h} \qquad \dot q_h = \frac{\partial H}{\partial p_h}$$

Parentesi di Poisson

$$\{f, g\} = \underbrace{\sum_{i,j=1}^{2n} \frac{\partial f}{\partial x_i} E_{ij} \frac{\partial g}{\partial x_j}} = \bar\nabla_x f \cdot E\,\bar\nabla_x g$$

$$E_{ij} = \begin{cases} 0 & i,j = 1,\dots,n \\ -\delta_{hk} & i = h,\ j = k+n \qquad h,k = 1,\dots,n \\ \delta_{hk} & i = h+n,\ j = k \qquad h,k = 1,\dots,n \\ 0 & i,j = n+1,\dots,2n \end{cases}$$

$$\{f, g\} = \sum_k \left[\frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k}\right]$$

Proprietà delle Parentesi di Poisson

a) $\{f, g\} = -\{g, f\} \Rightarrow \{f, f\} = 0$ ANTISIMMETRICA

b) $\{f, \underbrace{\alpha_1 g_1 + \alpha_2 g_2}_{\text{combinaz. lineare}}\} = \alpha_1 \{f, g_1\} + \alpha_2 \{f, g_2\}$ BILINEARE

c) $\{f, g_1 \cdot g_2\} = \{f, g_1\} g_2 + g_1 \{f, g_2\}$

d) Identità di Jacobi

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

La Par. di Poisson è un'applicazione BILINEARE ANTISIMMETRICA che SODDISFA L'ID. DI JACOBI (ci permette di definire un'ALGEBRA di LIE)

Dim.

$$\{f, g\} = \sum_{h=1}^{m} \left[\frac{\partial f}{\partial q_h} \frac{\partial g}{\partial p_h} - \frac{\partial f}{\partial p_h} \frac{\partial g}{\partial q_h} \right] = \sum_{i,j=1}^{2m} \frac{\partial f}{\partial x_i} E_{ij} \frac{\partial g}{\partial x_j}$$

a)
b) } ovvie

c) $\sum_{ij} \frac{\partial f}{\partial x_i} E_{ij} \overbrace{\frac{\partial}{\partial x_j} (g_1 \cdot g_2)}^{\partial g_1 \cdot g_2 + g_1 \cdot \partial g_2} = \left(\sum_{ij} \frac{\partial f}{\partial x_i} E_{ij} \frac{\partial g_1}{\partial x_j} \right) g_2 + g_1 \left(\sum_{i,j} \frac{\partial f}{\partial x_i} E_{ij} \frac{\partial g_2}{\partial x_j} \right)$

se $F(x,y) = x$

$\frac{\partial F}{\partial y} = 0$

$\frac{\partial F}{\partial x} = 1$

e) PARENTESI DI POISSON FONDAMENTALI:

$$\{p_h, p_k\} = \sum_{l=1}^{m} \left[\frac{\partial p_h}{\partial q_l} \frac{\partial p_k}{\partial p_l} - \frac{\partial p_h}{\partial p_l} \frac{\partial p_k}{\partial q_l} \right] = 0$$

Intesa come la funzione t.c.

$(\bar{p}, \bar{q}) \longmapsto p_h$

$$\{q_h, q_k\} = 0$$

$$\{p_h, q_k\} = \sum_{l=1}^{m} \left[\cancel{\frac{\partial p_h}{\partial q_l}}^{=0} \frac{\partial q_k}{\partial p_l} - \underbrace{\frac{\partial p_h}{\partial p_l}}_{\delta_{hl}} \underbrace{\frac{\partial q_k}{\partial q_l}}_{\delta_{kl}} \right] = -\sum_l \delta_{hl} \delta_{kl} = -\delta_{hk}$$

$$\{ X_i , X_j \} = E_{ij}$$