Definition 1.5.8 Relative Entropy

Let $\rho_1, \rho_2 \in \mathcal{S}(s)$ such that $\ker(\rho_2) \subseteq \ker(\rho_1)$.

Then, the relative entropy of ρ_1 with respect to ρ_2 is

$$\boxed{S(\rho_1 | \rho_2) := \operatorname{Tr} \rho_1 (\log \rho_1 - \log \rho_2)}$$

Lemma 1.5.9 $S(\rho_1 | \rho_2) \geq 0$

Proof:

1. if $\boxed{\sum_k c_k f_k(x) g_k(y) \geq 0}$ for $c_k \in \mathbb{R}$, $f_k, g_k : [a,b] \longrightarrow \mathbb{R}$

then $\boxed{\sum_k c_k \operatorname{Tr}(f_k(A) g_k(B)) \geq 0}$ for A, B with spectrum in $[a,b]$

$$\sum_k c_k \operatorname{Tr}(f_k(A) g_k(B)) = \sum_k c_k f_k(a_r) g_k(b_s) \operatorname{Tr}(P_r Q_s) \geq 0$$

for $A = \sum_r a_r P_r$ and $B = \sum_s b_s Q_s$ (spectral representations)

- for $\eta(x) = -x\log x$, $\eta(x) - \eta(y) \leq \eta'(y)(x-y)$ (concavity)

  $x, y \in [0,1]$

$$\eta(y) - \eta(x) \geq -(1 + \log y)(y - x) \quad \text{implies}$$

$$\mathrm{Tr}(\eta(\rho_2)) - \mathrm{Tr}(\eta(\rho_1)) \geq -\mathrm{Tr}\big((1 + \log\rho_2)(\rho_2 - \rho_1)\big)$$

$$= \mathrm{Tr}(\eta(\rho_2)) + \mathrm{Tr}\,\rho_1 \log\rho_2$$

$$\boxed{\mathrm{Tr}\ \rho_1(\log\rho_1 - \log\rho_2) = S(\rho_1|\rho_2) \geq 0}$$

Proposition 1.5.3 <u>Subadditivity of $S(\rho)$</u>

Let $\rho_{12} \in \mathcal{S}(S_1 + S_2)$ be a bipartite state with marginal states $\rho_1 = \mathrm{Tr}_2\,\rho_{12}$ and $\rho_2 = \mathrm{Tr}_1\,\rho_{12}$; then, $\boxed{S(\rho_{12}) \leq S(\rho_1) + S(\rho_2)}$

<u>Proof</u> : $0 \leq S(\rho_{12} | \rho_1 \otimes \rho_2) = \mathrm{Tr}\left(\rho_{12}\left(\log \rho_{12} - \log(\rho_1 \otimes \rho_2)\right)\right)$

set $\rho_1 = \sum_{i \in I_1} r_{1i} P_{1i}$, $\rho_2 = \sum_{i \in I_2} r_{2i} P_{2i}$ (spectral representations)

$$\log(\rho_1 \otimes \rho_2) = \sum_{i \in I_1, j \in I_2} \log(r_{1i} r_{2j}) P_{1i} \otimes P_{2j}$$

$$= \sum_{i \in I_1, j \in I_2} (\log r_{1i} + \log r_{2j}) P_{1i} \otimes P_{2j} = \log \rho_1 \otimes 1_2 + 1_1 \otimes \log \rho_2$$

Then, $0 \leq \mathrm{Tr}(\rho_{12} \log \rho_{12}) - \mathrm{Tr}\left(\rho_{12}(\log \rho_1) \otimes 1_2\right) - \mathrm{Tr}\left(\rho_{12} 1_1 \otimes (\log \rho_2)\right)$

$$0 \leq -S(\rho_{12}) + S(\rho_1) + S(\rho_2)$$

$\boxed{\text{Corollary 1.5.1}}$ $S(\rho_{12}) = S(\rho_1) + S(\rho_2) \iff \rho_{12} = \rho_1 \otimes \rho_2$

<u>Proof</u> $S(\rho_1 | \rho_2) = 0 \iff \rho_1 = \rho_2$

1.6. von Neumann entropy and bipartite entanglement

Observation : consider two qubits in the entangled state

$$|\psi^{(2)}_{unif}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \quad , \quad P^{(2)}_{unif} = |\psi^{(2)}_{unif}\rangle\langle\psi^{(2)}_{unif}|$$

This pure state has zero entropy :

$\boxed{S(P^{(2)}_{unif}) = 0}$, none the less its two marginals

$$\rho_1 = \mathrm{Tr}_2\, P^{(2)}_{unif} = \frac{1}{2} = \rho_2 = \mathrm{Tr}_1\, P^{(2)}_{unif}$$ have both

maximal entropies

$$\boxed{S[\rho_1) = S(\rho_2) = \log 2}$$

The whole (compound system $S_1 + S_2$) can be perfectly known,

$\boxed{S(P^{(2)}_{unif}) = 0}$, without knowing anything of its parties,

$$\boxed{S[\rho_1] = S(\rho_2) = \log 2}$$

<u>Remark</u> : This phenomenon is in contrast to the classical relations, $H(X,Y) \geq \max(H(X), H(Y))$, that says that the knowledge of the whole is always greater than the knowledge of its parts.
This quantum occurrence $S(\rho_{12}) - S(\rho_1) \ngeq 0$ is due to entanglement.

$\boxed{\text{Lemma 1.6.1}}$ Let $|\psi_{12}\rangle$ be a separable state vector of a bipartite system $S_1 + S_2$. Then,

$S(P_{12}) = S(P_1) = S(P_2) = 0$, where $P_{12} = |\psi_{12}\rangle\langle\psi_{12}|$

and $P_1 = \mathrm{Tr}_2 P_{12}$, $P_2 = \mathrm{Tr}_1 P_{12}$.

<u>Proof.</u> $|\psi_{12}\rangle$ is separable iff $|\psi_{12}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$,

then P_{12}, $P_1 = |\psi_1\rangle\langle\psi_1|$ and $P_2 = |\psi_2\rangle\langle\psi_2|$ are pure states with zero von Neumann entropy.

What about density matrices?

$\boxed{\text{Lemma 1.6.2}}$ $\quad \rho_{12} \in \mathcal{S}(S_1+S_2)$ separable $\Longrightarrow S(\rho_{12}) \geq \max\{S(\rho_1), S(\rho_2)\}$

Proof: assume $\rho_{12} = \sum_{i,j} \lambda_{ij}\, \rho_i^{(1)} \otimes \rho_j^{(2)}$ so that

$$\rho_1 = \mathrm{Tr}_2\, \rho_{12} = \sum_i \Big(\sum_j \lambda_{ij}\Big) \rho_i^{(1)} \quad , \quad \rho_2 = \mathrm{Tr}_1\, \rho_{12} = \sum_j \Big(\sum_i \lambda_{ij}\Big) \rho_j^{(2)} .$$

Set $d_2 = \dim(\mathcal{H}_2)$, then $\boxed{S(\rho_{12}, \rho_1 \otimes \frac{1}{d_2}) \geq 0}$.

$$0 \leq S\Big(\rho_{12}, \rho_1 \otimes \frac{1}{d_2}\Big) = S\Big(\sum_{i,j} \lambda_{ij}\, \rho_i^{(1)} \otimes \rho_j^{(2)} \Big| \sum_{i,j} \lambda_{ij}\, \rho_i^{(1)} \otimes \frac{1}{d_2}\Big)$$

$$\leq \sum_{i,j} \lambda_{ij}\, S\Big(\rho_i^{(1)} \otimes \rho_j^{(2)} \Big| \rho_i^{(1)} \otimes \frac{1}{d_2}\Big) \qquad \text{(Joint convexity of } S(\rho_1|\rho_2)\text{)}$$

$$= \sum_{i,j} \lambda_{ij}\, \mathrm{Tr}_{12}\Big(\rho_i^{(1)} \otimes \rho_j^{(2)} \big(\log \rho_i^{(1)} + \log \rho_j^{(2)} - \log \rho_i^{(1)} + \log d_2\big)\Big)$$

$$= \sum_{i,j} \lambda_{ij}\, \mathrm{Tr}_2\, \rho_j^{(2)} \log \rho_j^{(2)} + \log d_2 = \log d_2 - \sum_{i,j} \lambda_{ij}\, S(\rho_j^{(2)})$$

$$S(\rho_{12} \,|\, \rho_1 \otimes \frac{1}{d_2}) = \mathrm{Tr}_{12}\, \rho_{12} \left(\log \rho_{12} - \log \rho_1 + \log d_2 \right) = - S(\rho_{12}) + S(\rho_1) + \log d_2$$

(from page 81) $\leq \log d_2 - \sum_{i,j} d_{ij}\, S(\rho_j^{(2)})$

Then, $S(\rho_{12}) \geq S(\rho_1) + \sum_{i,j} d_{ij}\, S(\rho_j^{(2)}) \geq S(\rho_1)$

Remark: the entropy of a bipartite separable states
(uncertainty about the whole compound system)
is always larger than or equal to the entropy
of the marginal density matrices
(uncertainty about the constituent parties).

Question: is the von Neumann entropy an exhaustive entanglement witness? Namely, if

$S(\rho_{12}) \geq \max\{S(\rho_1), S(\rho_2)\}$ is then ρ_{12} separable?

$$\boxed{NO}$$

Example 1.6.1 Horodecky's states

Let us consider the family of states $\rho_F \in M_d(\mathbb{C}) \otimes M_d(\mathbb{C})$ of the form

$$\boxed{\rho_F = a\mathbb{1} + b\,P^{(d)}_{unif}} \quad \text{where } P^{(d)}_{unif} = \frac{1}{d}\sum_{i,j=1}^{d} |i\rangle\langle j| \otimes |i\rangle\langle j|,\ \{|i\rangle\}_{i=1}^{d} \text{ ONB in } \mathbb{C}^d$$

$$\begin{cases} \mathrm{Tr}\,\rho_F = 1 = a d^2 + b \\ F := \mathrm{Tr}(\rho_F P^{(d)}_{unif}) = \mathrm{Tr}(a P^{(d)}_{unif} + b\, P^{(d)}_{unif}) = a + b \geq 0 \end{cases} \Longrightarrow \begin{cases} 1 = a d^2 + F - a \\ b = F - a \end{cases} \Longrightarrow \begin{cases} a = \dfrac{1-F}{d^2-1} \\ b = \dfrac{F d^2 - 1}{d^2-1} \end{cases}$$

Positivity of ρ_F: $\quad \mathrm{Spectrum}(\rho_F) = \begin{cases} a \geq 0 \\ a + b = F \geq 0 \end{cases} \Longrightarrow \boxed{0 \leq F \leq 1}$

Partial transposition :

$$\rho_F = \frac{1-F}{d^2-1} + \frac{Fd^2-1}{d^2-1} P^{(d)}_{unif} \quad \text{sent into}$$

$$\rho'_F = \frac{1-F}{d^2-1} + \frac{Fd^2-1}{d^2-1}\frac{1}{d} V \ , \quad V|\psi\otimes\phi\rangle = |\phi\otimes\psi\rangle$$

$$\text{Spectrum}(\rho'_F) = \left\{ \frac{1-F}{d^2-1} \pm \frac{Fd^2-1}{d^2-1}\frac{1}{d} \right\} = \left\{ \frac{d - dF \pm Fd^2 \mp 1}{d(d^2-1)} \right\} = \left\{ \frac{\pm Fd(d\mp 1) + d \mp 1}{d(d+1)(d-1)} \right\}$$

$$= \left\{ \frac{Fd+1}{d(d+1)} , \frac{1-Fd}{d(d-1)} \right\}$$

ρ_F is Positive under Partial Transposition (PPT : see Section 1.4.)

if $1-Fd \geq 0 \iff 0 \leq F \leq \frac{1}{d}$.

If $F > \frac{1}{d}$ we know ρ_F is entangled.

For such a class of states it is also known that PPT is equivalent to separability

What about the von Neumann entropy?

$$S(\rho_F) = -(d^2-1)\,\frac{1-F}{d^2-1}\log\frac{1-F}{d^2-1} - F\log F$$

$$\rho_F^{(1)} := \mathrm{Tr}_2\,\rho_F = d\,\frac{1-F}{d^2-1}\,\mathbb{1}_1 + \frac{Fd^2-1}{d^2-1}\,\frac{1}{d}\,\mathbb{1}_1 = \frac{d^2-Fd^2+Fd^2-1}{d(d^2-1)} = \frac{\mathbb{1}_1}{d}$$

$$\rho_F^{(2)} := \mathrm{Tr}_1\,\rho_F = \frac{d(1-F)}{d^2-1}\,\mathbb{1}_2 + \frac{Fd^2-1}{d^2-1}\,\frac{1}{d}\,\mathbb{1}_2 = \frac{\mathbb{1}_2}{d}$$

Choose $d=2$: $\quad S(\rho_F) = -(1-F)\log\frac{1-F}{3} - F\log F$

$$S(\rho_F^{(1)}) = S(\rho_F^{(2)}) = \log 2$$

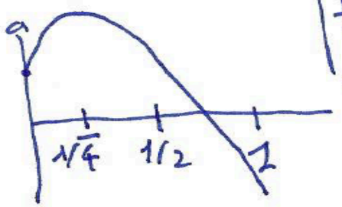
$$\boxed{S(\rho_F) - S(\rho_F^{(1)}) = -(1-F)\log\frac{1-F}{3} - F\log F - \log 2} = f(F)$$

$f(0) = \log 3 - \log 2 > 0 \qquad f(1) = -\log 2$

$f'(F) = \log\frac{1-F}{3F}$

$f''(F) = -\frac{1}{F(1-F)}$

$f(F)$



$$\boxed{f\left(\tfrac{1}{2}\right) = \log\frac{3}{2} > 0} \Rightarrow \boxed{\exists\; F > \tfrac{1}{2} \text{ with } \begin{cases} \rho_F \text{ entangled} \\ S(\rho_F) > S(\rho_F^{(1)}) \end{cases}}$$