LESSON 14.

1. Product of quasi–projective varieties and tensors.

1.1. **Products.** In Lesson 3, Section 1.5, we have seen how the product $\mathbb{P}^1 \times \mathbb{P}^1$ can be interpreted as a projective variety, and precisely a quadric of maximal rank, by means of the Segre map. Now we want to give a structure of algebraic variety to all products of algebraic varieties. We will seen that this can be done by generalizing the definition of the Segre map to any product of projective spaces $\mathbb{P}^n \times \mathbb{P}^m$.

Let $\mathbb{P}^n$, $\mathbb{P}^m$ be projective spaces over the same field K. The cartesian product $\mathbb{P}^n \times \mathbb{P}^m$ is simply a set: we want to define an injective map from $\mathbb{P}^n \times \mathbb{P}^m$ to a suitable projective space, so that the image is a projective variety, which will be identified with our product.

Let $N = (n+1)(m+1) - 1$ and define $\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ in the following way: $\sigma([x_0, \ldots, x_n], [y_0, \ldots, y_m]) = [x_0y_0, x_0y_1, \ldots, x_iy_j, \ldots, x_ny_m]$. Using coordinates w_{ij}, $i = 0, \ldots, n$, $j = 0, \ldots, m$, in $\mathbb{P}^N$, σ is defined by

$$\{w_{ij} = x_iy_j, \ i = 0, \ldots, n, j = 0, \ldots, m.$$

It is easy to observe that σ is a well–defined map.

Let $\Sigma_{n,m}$ (or simply Σ) denote the image $\sigma(\mathbb{P}^n \times \mathbb{P}^m)$.

Proposition 1.1. *σ is injective and $\Sigma_{n,m}$ is a closed subset of $\mathbb{P}^N$.*

Proof. If $\sigma([x], [y]) = \sigma([x'], [y'])$, then there exists $\lambda \neq 0$ such that $x'_iy'_j = \lambda x_iy_j$ for all i, j. In particular, if $x_h \neq 0$, $y_k \neq 0$, then also $x'_h \neq 0$, $y'_k \neq 0$, and for all i $x'_i = \lambda \frac{y_k}{y'_k} x_i$, so $[x_0, \ldots, x_n] = [x'_0, \ldots, x'_n]$. Similarly for the second point.
To prove the second assertion, I claim: $\Sigma_{n,m}$ is the closed set of equations:

$$\{w_{ij}w_{hk} = w_{ik}w_{hj}, \ i, h = 0, \ldots, n; j, k = 0 \ldots, m. \tag{1}$$

It is clear that if $[w_{ij}] \in \Sigma$, then it satisfies (1).

Conversely, assume that $[w_{ij}]$ satisfies (1) and that $w_{\alpha\beta} \neq 0$. Then

$$\begin{aligned} [w_{00}, \ldots, w_{ij}, \ldots, w_{nm}] &= [w_{00}w_{\alpha\beta}, \ldots, w_{ij}w_{\alpha\beta}, \ldots, w_{nm}w_{\alpha\beta}] = \\ &= [w_{0\beta}w_{\alpha 0}, \ldots, w_{i\beta}w_{\alpha j}, \ldots, w_{n\beta}w_{\alpha m}] = \\ &= \sigma([w_{0\beta}, \ldots, w_{n\beta}], [w_{\alpha 0}, \ldots, w_{\alpha m}]). \end{aligned}$$

□

σ is called the Segre map and $\Sigma_{n,m}$ the Segre variety or biprojective space. Note that Σ is covered by the affine open subsets $\Sigma^{ij} = \Sigma \cap W_{ij}$, where $W_{ij} = \mathbb{P}^N \setminus V_P(w_{ij})$. Moreover $\Sigma^{ij} = \sigma(U_i \times V_j)$, where $U_i \times V_j$ is naturally identified with $\mathbb{A}^{n+m}$.

Proposition 1.2. $\sigma|_{U_i \times V_j} : U_i \times V_j = \mathbb{A}^{n+m} \to \Sigma^{ij}$ *is an isomorphism of varieties.*

Proof. Assume by simplicity $i = j = 0$. Choose non–homogeneous coordinates on U_0: $u_i = x_i/x_0$ and on V_0: $v_j = y_j/y_0$. So $u_1, \dots u_n, v_1, \dots, v_m$ are coordinates on $U_0 \times V_0$. Take non–homogeneous coordinates also on W_{00}: $z_{ij} = w_{ij}/w_{00}$.

Using these coordinates we have:

$$\begin{aligned}\sigma|_{U_i \times V_j} :&(u_1, \dots u_n, v_1, \dots, v_m) \to (v_1, \dots, v_m, u_1, u_1v_1, \dots, u_1v_m, \dots, u_nv_m)\\ &\qquad\qquad ||\\ &([1, u_1, \dots, u_n], [1, v_1, \dots, v_m])\end{aligned}$$

i.e. $\sigma(u_1, \dots, v_m) = (z_{01}, \dots, z_{nm})$, where

$$\begin{cases} z_{i0} = u_i, & \text{if } i = 1, \dots, n; \\ z_{0j} = v_j, & \text{if } j = 1, \dots, m; \\ z_{ij} = u_iv_j = z_{i0}z_{0j} & \text{otherwise.} \end{cases}$$

Hence $\sigma|_{U_0 \times V_0}$ is regular.
The inverse map takes $(z_{01}, \dots, z_{nm})$ to $(z_{10}, \dots, z_{n0}, z_{01}, \dots, z_{0m})$, so it is also regular. □

Corollary 1.3. $\mathbb{P}^n \times \mathbb{P}^m$ *is irreducible and birational to* $\mathbb{P}^{n+m}$.

Proof. The first assertion follows from Ex.5, Lesson 7, considering the covering of Σ by the open subsets Σ^{ij}. Indeed, $\Sigma^{ij} \cap \Sigma^{hk} = \sigma((U_i \times V_j) \cap (U_h \times V_k)) = \sigma((U_i \cap U_h) \times (V_j \cap V_k))$, and $U_i \cap U_h \neq \emptyset \neq V_j \cap V_k$.

For the second assertion, by Theorem 1.6, Lesson 13, it is enough to note that $\Sigma_{n,m}$ and $\mathbb{P}^{n+m}$ contain isomorphic open subsets, i.e. Σ^{ij} and $\mathbb{A}^{n+m}$. □

From now on, we shall identify $\mathbb{P}^n \times \mathbb{P}^m$ with $\Sigma_{n,m}$. If $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$ are any quasi–projective varieties, then $X \times Y$ will be automatically identified with $\sigma(X \times Y) \subset \Sigma$.

Proposition 1.4. *If* X *and* Y *are projective varieties (resp. quasi–projective varieties), then* $X \times Y$ *is projective (resp. quasi–projective).*

Proof.

$$\begin{aligned}\sigma(X\times Y) &= \bigcup_{i,j}(\sigma(X\times Y)\cap\Sigma^{ij}) = \\ &= \bigcup_{i,j}(\sigma(X\times Y)\cap(U_i\times V_j)) = \\ &= \bigcup_{i,j}(\sigma((X\cap U_i)\times(Y\cap V_j))).\end{aligned}$$

If X and Y are projective varieties, then $X\cap U_i$ is closed in U_i and $Y\cap V_j$ is closed in V_j, so their product is closed in $U_i\times V_j$; since $\sigma|_{U_i\times V_j}$ is an isomorphism, also $\sigma(X\times Y)\cap\Sigma^{ij}$ is closed in Σ^{ij}, so $\sigma(X\times Y)$ is closed in Σ, by Lemma 1.3, Lesson 10.

If X, Y are quasi–projective, the proof is similar: $X\cap U_i$ is locally closed in U_i and $Y\cap V_j$ is locally closed in V_j, so $X\cap U_i = Z\setminus Z'$, $Y\cap V_j = W\setminus W'$, with Z, Z', W, W' closed. Therefore $(Z\setminus Z')\times(W\setminus W') = Z\times W\setminus((Z'\times W)\cup(Z\times W'))$, which is locally closed.

As for the irreducibility, see Exercise 1, this Lesson. □

Example 1.5. $\mathbb{P}^1\times\mathbb{P}^1$

The example of $\mathbb{P}^1\times\mathbb{P}^1$, the Segre quadric, has already been studied in Lesson 3, 1.5.

We recall that $\sigma:\mathbb{P}^1\times\mathbb{P}^1\to\mathbb{P}^3$ is given by the parametric equations $\{w_{ij}=x_iy_j,\ i=0,1,$ $j=0,1$. Σ has only one non–trivial equation: $w_{00}w_{11}-w_{01}w_{10}$, hence Σ is a quadric. The equation of Σ can be written as

$$\begin{vmatrix} w_{00} & w_{01} \\ w_{10} & w_{11}\end{vmatrix} = 0. \tag{2}$$

Σ contains two families of special closed subsets parametrised by $\mathbb{P}^1$, i.e.

$$\{\sigma(\{P\}\times\mathbb{P}^1)\}_{P\in\mathbb{P}^1} \quad\text{and}\quad \{\sigma(\mathbb{P}^1\times\{Q\})\}_{Q\in\mathbb{P}^1}.$$

If $P=[a_0,a_1]$, then $\sigma(\{P\}\times\mathbb{P}^1)$ is given by the equations:

$$\begin{cases} w_{00}=a_0y_0\\ w_{01}=a_0y_1\\ w_{10}=a_1y_0\\ w_{11}=a_1y_1\end{cases}$$

hence it is a line. Cartesian equations of $\sigma(\{P\}\times\mathbb{P}^1)$ are:

$$\begin{cases} a_1w_{00}-a_0w_{10}=0\\ a_1w_{01}-a_0w_{11}=0;\end{cases}$$

they express the proportionality of the rows of the matrix (2) with coefficients $[a_1, -a_0]$. Similarly, $\sigma(\mathbb{P}^1 \times \{Q\})$ is the line of equations

$$\begin{cases} a_1 w_{00} - a_0 w_{01} = 0 \\ a_1 w_{10} - a_0 w_{11} = 0. \end{cases}$$

Hence Σ contains two families of lines, called the rulings of Σ: two lines of the same ruling are clearly disjoint, while two lines of different rulings intersect at one point $(\sigma(P, Q))$. Conversely, through any point of Σ there pass two lines, one for each ruling.

Note that Σ is exactly the quadric surface of Lesson 13, Example 1.9.d) and that the projection π_P of centre $P[1, 0, 0, 0]$ realizes an explicit birational map between $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$. The two lines contained in Σ passing through P have equations $w_{10} = w_{11} = 0$ and $w_{01} = w_{11} = 0$ respectively; they are contracted to the points $E_0[1, 0, 0]$, $E_1[0, 1, 0]$ of $\mathbb{P}^2$ respectively. Conversely, the line $x_2 = 0$ in $\mathbb{P}^2$ passing through E_0, E_1 is contracted to P by π_p^{-1}.

1.2. **Tensors.** The product of projective spaces has a coordinate-free description in terms of tensors. Precisely, let $\mathbb{P}^n = \mathbb{P}(V)$ and $\mathbb{P}^m = \mathbb{P}(W)$. The tensor product $V \otimes W$ of the vector spaces V, W is constructed as follows: let $K(V \times W)$ be the K-vector space with basis $V \times W$ obtained as the set of formal finite linear combinations of type $\Sigma_i a_i(v_i, w_i)$ with $a_i \in K$. Let U be the vector subspace generated by all elements of the form:

$(v, w) + (v', w) - (v + v', w)$,
$(v, w) + (v, w') - (v, w + w')$,
$(\lambda v, w) - \lambda(v, w)$,
$(v, \lambda w) - (\lambda(v, w)$,

with $v, v' \in V$, $w, w' \in W$, $\lambda \in K$. The tensor product is by definition the quotient $V \otimes W := K(V \times W)/U$. The class of a pair (v, w) is denoted by $v \otimes w$, and called a decomposable tensor. So $V \otimes W$ is generated by the decomposable tensors; more precisely, a general element $\omega \in V \otimes W$ is of the form $\Sigma_{i=1}^k v_i \otimes w_i$, with $v_i \in V$, $w_i \in W$. The minimum k such that an expression of this type exists is called the tensor rank of ω.

There is a natural bilinear map $\otimes : V \times W \to V \otimes W$, such that $(v, w) \to v \otimes w$. It enjoys the following universal property: for any K-vector space Z with a bilinear map $f : V \times W \to Z$, there exists a unique linear map $\bar{f} : V \otimes W \to Z$ such that f factorizes in the form $f = \bar{f} \circ \otimes$.

If $\dim V = n + 1$, $\dim W = m + 1$, and bases $\mathcal{B} = (e_0, \ldots, e_n), \mathcal{B}' = (e'_0, \ldots, e'_m)$ are given, then $(e_0 \otimes e'_0, \ldots, e_i \otimes e'_j, \ldots e_n \otimes e'_m)$ is a basis of $V \otimes W$: therefore $\dim V \otimes W = (n+1)(m+1)$.

If $v = x_0 e_0 + \cdots + x_n e_n$, $w = y_0 e'_0 + \cdots + y_m e'_m$, then $v \otimes w = \sum_{i,j} x_i y_j e_i \otimes e'_j$. So, passing to the projective spaces, the map $\otimes$ defines precisely the Segre map

$$\sigma : \mathbb{P}(V) \times \mathbb{P}(W) \to \mathbb{P}(V \otimes W), \quad ([v], [w]) \to [v \otimes w].$$

Indeed in coordinates we have $([x_0,\dots,x_n],[y_0,\dots,y_m]) \to [w_{00},\dots,w_{nm}]$, with $w_{ij}=x_iy_j$. The image of $\otimes$ is the set of decomposable tensors, or rank one tensors.

The tensor product $V\otimes W$ has the same dimension, and is therefore isomorphic to the vector space of $(n+1)\times(m+1)$ matrices. The coordinates w_{ij} can be interpreted as the entries of such a $(n+1)\times(m+1)$ matrix. The equations of the Segre variety $\Sigma_{n,m}$ are the 2×2 minors of the matrix, therefore $\Sigma_{n,m}$ can be interpreted as the set of matrices of rank one.

The construction of the tensor product can be iterated, to construct $V_1\otimes V_2\otimes\cdots\otimes V_r$. The following properties can easily be proved:

1. $V_1\otimes(V_2\otimes V_3)\simeq(V_1\otimes V_2)\otimes V_3$;
2. $V\otimes W\simeq W\otimes V$;
3. $V^*\otimes W\simeq Hom(V,W)$, where $f\otimes w\to(V\to W: v\to f(v)w)$.

Also the Veronese morphism has a coordinate free description, in terms of symmetric tensors. Given a vector space V, for any $d\geq 0$ the d-th symmetric power of V, S^dV or Sym^dV, is constructed as follows. We consider the tensor product of d copies of V: $V\otimes\cdots\otimes V=V^{\otimes d}$, and we consider its subvector space U generated by all tensors of the form $v_1\otimes\dots v_d - v_{\sigma(1)}\otimes\dots\otimes v_{\sigma(d)}$, where $v_1,\dots,v_d$ vary in V and σ varies in the symmetric group on d elements $\mathcal{S}_d$. Then by definition $S^dV := V^{\otimes d}/U$. The equivalence class $[v_1\otimes\cdots\otimes v_d]$ is denoted as a product $v_1\dots v_d$.

There is a natural multilinear and symmetric map $V\times\cdots\times V=V^d\to S^dV$, such that $(v_1,\dots,v_d)\to v_1\dots v_d$, which enjoys the universal property. S^dV is generated by the products $v_1\dots v_d$.

In characteristic 0, S^dV can also be interpreted as a subspace of $V^{\otimes d}$, by considering the following map, that is an isomorphism to the image:

$$S^dV\to V^{\otimes d},\quad v_1\dots v_d\to\Sigma_{\sigma\in\mathcal{S}_d}\frac{1}{d!}v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(d)}.$$

For instance, in S^2V the product v_1v_2 can be identified with $\frac{1}{2}(v_1\otimes v_2+v_2\otimes v_1)$.

If $\mathcal{B}=(e_0,\dots,e_n)$ is a basis of V, then it is easy to check that a basis of S^dV is formed by the monomials of degree d in $e_0,\dots,e_n$; therefore $\dim S^dV=\binom{n+d}{d}$.

The symmetric algebra of V is $SV:=\oplus_{d\geq 0}S^dV=K\oplus V\oplus S^2V\oplus\dots$. An inner product can be naturally defined to give it the structure of a K-algebra, which results to be isomorphic to the polynomial ring in $n+1$ variables, where $n+1=\dim V$.

If $char K = 0$ the Veronese morphism can be interpreted in the following way:

$$v_{n,d} : \mathbb{P}(V) \to \mathbb{P}(S^d V),\ [v] = [x_0 e_0 + \dots x_n e_n] \to [v^d] = [(x_0 e_0 + \dots + x_n e_n)^d].$$

Moreover $S^2 V$ can be interpreted as the space of symmetric $(n+1) \times (n+1)$ matrices, and the Veronese variety $V_{n,2}$ as the subset of the symmetric matrices of rank one, because its equations express precisely the vanishing of the minors of order 2 (see Example 1.13, (2), Lesson 11).

Exercises 1.6. 1. Using Ex. 5 of Lesson 7, prove that, if $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$ are irreducible projective varieties, then $X \times Y$ is irreducible.

2. Let L, M, N be the following lines in $\mathbb{P}^3$:

$$L : x_0 = x_1 = 0, M : x_2 = x_3 = 0, N : x_0 - x_2 = x_1 - x_3 = 0.$$

Let X be the union of lines meeting L, M and N: write equations for X and describe it: is it a projective variety? If yes, of what dimension and degree?

3. Let X, Y be quasi–projective varieties, identify $X \times Y$ with its image via the Segre map. Check that the two projection maps $X \times Y \xrightarrow{p_1} X$, $X \times Y \xrightarrow{p_2} Y$ are regular. (Hint: use the open covering of the Segre variety by the Σ^{ij}'s.)