PARENTESI DI POISSON

Date $f, g : \mathbb{R}^{2n+1} \to \mathbb{R}$, funz. di $(\bar{p}, \bar{q}, t) = (\bar{x}, t)$

$$\{f, g\} = \sum_{l=1}^{m} \left[\frac{\partial f}{\partial q_l} \frac{\partial g}{\partial p_l} - \frac{\partial f}{\partial p_l} \frac{\partial g}{\partial q_l} \right] = \sum_{i,j=1}^{2m} \frac{\partial f}{\partial x_i} E_{ij} \frac{\partial g}{\partial x_j}$$

$$E = \begin{pmatrix} 0 & -\mathbb{1}_m \\ \mathbb{1}_m & 0 \end{pmatrix}, \text{ cioè } E_{hk} = 0, \; E_{h,k+m} = -\delta_{hk}$$

$$E_{h+m,k} = \delta_{hk}, \; E_{h+m,k+m} = 0 \qquad h,k = 1,\dots,m$$

In particolare: Parentesi di Poisson fondamentali:

$$\{p_l, p_k\} = \{q_l, q_k\} = 0 \qquad l,k = 1,\dots,m$$

$$\{p_l, q_k\} = -\{q_k, p_l\} = -\delta_{kl}$$

$$\{x_r, x_s\} = E_{rs} \qquad r,s = 1,\dots,2m$$

$$= \sum_{i,j} \frac{\partial x_r}{\partial x_i} E_{ij} \frac{\partial x_s}{\partial x_j} = \sum_{ij} \delta_{ri} E_{ij} \delta_{js} \; //$$

La Par. di Poisson è un'applicazione BILINEARE ANTISIMMETRICA che SODDISFA l'identità di Jacobi

$$\underbrace{\{f, \{g, h\}\} + \{g, \{h, f\}\}} + \{h, \{f, g\}\} = 0$$

Notazione: def. $\partial_i f \equiv \frac{\partial f}{\partial x_i}$

Dim.

$$\{f, \{g, h\}\} - \{g, \{f, h\}\} =$$

$$= \sum_{ij=1}^{2m} \partial_i f \, E_{ij} \, \partial_j \left[\sum_{k,m=1}^{2m} \partial_k g \, E_{km} \, \partial_m h \right] - \sum_{a,b=1}^{2m} \partial_a g \, E_{ab} \, \partial_b \left[\sum_{c,d=1}^{2m} \partial_c f \, E_{cd} \, \partial_d h \right]$$

$$= \sum_{i,j,k,m} \left[\partial_i f \, E_{ij} \, \partial_j \partial_k g \, E_{km} \, \partial_m h + \underbrace{\partial_i f \, E_{ij} \, \partial_k g \, E_{km} \, \partial_j \partial_m h} \right]$$

$$- \sum_{a,b,c,d} \left[\partial_a g \, E_{ab} \, \partial_b \partial_c f \, E_{cd} \, \partial_d h + \underbrace{\partial_{\underset{k}{a}} g \, E_{ab} \, \partial_{\underset{i}{c}} f \, E_{cd} \, \partial_{\underset{m}{b}} \partial_{\underset{j}{d}} h}_{\substack{\text{Questo termine } (\sum_{abcd}) \text{ posso riscriverlo} \\ \text{rinominando gli indici sommati}}} \right]$$

$$= \sum_{ijkm} \partial_i f\, E_{ij}\, \partial_j \partial_k g\, E_{km}\, \partial_m h + \sum_{ijkm} \partial_i f\, E_{ij}\, \cancel{\partial_k g\, E_{km}}\, \partial_j \partial_m h$$

$$- \sum_{abcd} \underset{j}{\partial_a} g\, E_{ab}\, \underset{i}{\partial_b} \underset{k}{\partial_c} f\, \underset{k}{E_{cd}}\, \underset{m}{\partial_d} h - \sum_{ijkm} \cancel{\partial_k g\, E_{km}}\, \partial_i f\, E_{ij}\, \underbrace{\partial_m \partial_j h}_{\partial_j \partial_m h}$$

$$= \sum_{ijkm} \left[\partial_i f\, E_{ij}\, \partial_j \partial_k g\, E_{km}\, \partial_m h - \partial_j g\, \underbrace{E_{ji}}_{= -E_{ij}}\, \partial_i \partial_k f\, E_{km}\, \partial_m h \right]$$

$$= \sum_{ijkm} \left[\partial_i f\, E_{ij}\, \partial_k \partial_j g \left(E_{km}\, \partial_m h \right) + \partial_k \partial_i f\, E_{ij}\, \partial_j g \left(E_{km}\, \partial_m h \right) \right]$$

$$= \sum_{ijkm} \partial_k \left[\partial_i f\, E_{ij}\, \partial_j g \right] E_{km}\, \partial_m h =$$

$$= \sum_{km} \partial_k \underbrace{\left[\sum_{ij} \partial_i f\, E_{ij}\, \partial_j g \right]}_{\{f,g\}} E_{km}\, \partial_m h = \{\{f,g\},h\} =$$

$$= -\{h, \{f,g\}\} \quad //$$

Osservazione:

$$\{x_s, g(\bar{x})\} = \sum_{ij} \frac{\partial x_s}{\partial x_i} E_{ij} \frac{\partial g}{\partial x_j} = \sum_{ij} \delta_{si} E_{ij} \frac{\partial g}{\partial x_j}$$

$$= \sum_j E_{sj} \frac{\partial g}{\partial x_j}$$

In particolare, se $g = H$ (Hamiltoniana)

$$\dot{\bar{x}} = E \bar{\nabla} H \qquad \dot{x}_k = \sum_j E_{kj} \frac{\partial H}{\partial x_j} = \{x_k, H\}$$

Eq. Ham: $\dot{x}_i = \{x_i, H\}$ $\quad i = 1, \dots, 2n$ $\quad \leftarrow$ Es. di: $\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$

In termini di $\bar{p}$ e $\bar{q}$

$$\{p_h, g(\bar{p},\bar{q})\} = \sum_{j=1}^{2n} \underset{\substack{\uparrow \\ h=1,\dots,n}}{E_{hj}}^{\swarrow \neq 0 \text{ solo se } j=n+k,\, k=1,\dots,n \text{, e in questo caso } = -\delta_{hk}} \frac{\partial g}{\partial x_j} = \sum_{k=1}^{n} (-\delta_{hk}) \frac{\partial g}{\partial q_k} = -\frac{\partial g}{\partial q_h}$$

$$\{q_h, g(\bar{p},\bar{q})\} = \frac{\partial g}{\partial p_h}$$

PARENTESI di POISSON & MOMENTO ANGOLARE

Se prendiamo un corpo che si muove in $\mathbb{R}^3$, con coord cartesiane, allora $\bar{q}$ sono tali coordinate e le $\bar{p}$ mi danno le componenti della quantità di moto del corpo.

Le componenti del momento angolare sono funzioni di $\bar{p}$ e $\bar{q}$:

$$\bar{M} = \bar{q} \times \bar{p}$$

$$M_i = \sum_{m,k=1}^{3} \epsilon_{imk}\, q_m p_k$$

$$\{p_e, M_i\} = -\frac{\partial M_i}{\partial q_e} = -\frac{\partial}{\partial q_e} \sum_{mk} \epsilon_{imk}\, q_m p_k =$$

$$= -\sum_{mk} \epsilon_{imk}\, \delta_{me}\, p_k = -\sum_{k=1}^{3} \epsilon_{iek}\, p_k$$

$$= \{p_e, \sum_{mk} \epsilon_{imk}\, q_m p_k\} =$$

$$= \sum_{mk} \underbrace{\{p_e, q_m p_k\}}_{q_m \underset{0}{\underset{\shortparallel}{\{p_e,p_k\}}} + \underbrace{\{p_e, q_m\}}_{-\delta_{me}} p_k} \epsilon_{imk} = -\sum_{mk} \epsilon_{imk}\, \delta_{me}\, p_k = -\sum_{k=1}^{3} \epsilon_{iek}\, p_k$$

→ usando le parentesi di Poisson fondamentali e le proprietà delle parentesi di P., possiamo calcolare generiche parentesi senza dover fare derivate (cioè applicare la def. di partenza)

$$\{q_l, M_i\} = -\sum_m \epsilon_{ilm} q_m$$

Dimostriamo ora una relazione IMPORTANTE:

$$\{M_i, M_j\} = \sum_{k=1}^{3} \epsilon_{ijk} M_k$$

[Algebra di Lie su(2)]

$$\{M_i, M_j\} = \sum_{mh} \epsilon_{imh} \{q_m p_h, M_j\} =$$

$$= \sum_{mh} \epsilon_{imh} \Big[q_m \underbrace{\{p_h, M_j\}}_{= -\sum_s \epsilon_{jhs} p_s} + \underbrace{\{q_m, M_j\}}_{-\sum_s \epsilon_{jms} q_s} p_h \Big]$$

$$= -\sum_{mhs} \epsilon_{imh} \left[q_m p_s \epsilon_{jhs} + q_s p_h \epsilon_{jms} \right]$$

$$= -\sum_{ms} q_m p_s \underbrace{\sum_h \epsilon_{imh} \epsilon_{sjh}}_{\delta_{is}\delta_{mj} - \delta_{ij}\delta_{ms}} - \sum_{hs} q_s p_h \underbrace{\sum_m \epsilon_{ihm} \epsilon_{jsm}}_{\delta_{ij}\delta_{hs} - \delta_{is}\delta_{hj}}$$

$$= -q_j p_i + \delta_{ij} \cancel{\underbrace{\sum_m q_m p_m}_{\vec{q}\cdot\vec{p}}} - \delta_{ij} \cancel{\underbrace{\sum_h q_h p_h}_{\vec{q}\cdot\vec{p}}} + q_i p_j$$

$$= q_i p_j - q_j p_i$$

$$\sum_k \epsilon_{ijk} M_k = \sum_k \epsilon_{ijk} \sum_{lh} \underset{\underset{\epsilon_{lhk}}{\parallel}}{\epsilon_{klh}} q_l p_h = \sum_{lh} (\delta_{il}\delta_{jh} - \delta_{ih}\delta_{jl}) q_l p_h =$$

$$= q_i p_j - q_j p_i$$

//

$$\{M_i, M_j\} = \sum_k \epsilon_{ijk} M_k \Rightarrow \quad \{M_1, M_2\} = M_3 \quad \{M_2, M_3\} = M_1 \quad \{M_3, M_1\} = M_2$$

e tutte le altre $= 0$.

Parentesi di Poisson & Vettore di Runge-Lenz

$$\bar{A} = \bar{p} \times \bar{M} - k\frac{\bar{q}}{r} \qquad r(\bar{q}) = \sqrt{q_1^2 + q_2^2 + q_3^2}$$

(m=1)

$$H = \frac{\bar{p}^2}{2} - \frac{k}{r(\bar{q})} = \frac{1}{2}\sum_j p_j^2 - \frac{k}{r}$$

Verifichiamo che $\bar{A}$ è una cost. del moto, cioè verifichiamo che

$$\{A_i, H\} = 0$$

→ f cost. del moto se $\frac{\partial f}{\partial t} + \{f, H\} = 0$

($\bar{A}$ indip. da t esplicitam.)

$$A_i = \sum_{mh} \epsilon_{imh} p_m M_h - k\frac{q_i}{r}$$

$$\{A_i, H\} = \left\{\sum_{mh} \epsilon_{imh} p_m \sum_{rs} \epsilon_{hrs} q_r p_s - k\frac{q_i}{r}, \frac{1}{2}\sum_j p_j^2 - \frac{k}{r}\right\} =$$

$$\{f(\bar{q}), g(\bar{q})\} = 0$$

$$= \sum_{mhrs} \epsilon_{imh}\epsilon_{rsh} \left\{p_m q_r p_s, \cancel{\frac{1}{2}\sum_j p_j^2} - \frac{k}{r}\right\}$$

$$-\frac{k}{2}\left\{\frac{q_i}{r}, \sum_j p_j^2\right\}$$

$$\frac{\partial}{\partial p_r}\frac{1}{2}\sum_j p_j^2$$

$$\sum_{mhrs}\epsilon_{imh}\epsilon_{rsh}\left\{p_m q_r p_s, \frac{1}{2}\sum_j p_j^2\right\} = \sum_{m\,rs}(\delta_{ir}\delta_{ms}-\delta_{is}\delta_{mr})\,p_m p_s\underbrace{\left\{q_r, \frac{1}{2}\sum_j p_j^2\right\}}$$

$$= \sum_{m\,rs}(\delta_{ir}\delta_{ms}-\delta_{is}\delta_{mr})\,p_m p_s p_r = \bar{p}^2 p_i - \bar{p}^2 p_i = 0$$

$$\sum_{mhrs}\epsilon_{imh}\epsilon_{rsh}q_r\underbrace{\left\{p_m p_s, -\frac{k}{r}\right\}}_{p_m\left\{p_s, -\frac{k}{r}\right\} + p_s\left\{p_m, -\frac{k}{r}\right\}}$$

$$r = \sqrt{q_1^2+p_2^2+p_3^2}$$

$$\left\{p_e, \frac{1}{r}\right\} = -\frac{\partial}{\partial q_e}\frac{1}{r} = \frac{1}{r^2}\frac{\partial r}{\partial q_e} = \frac{1}{r^2}\frac{1}{2r}2q_e = \frac{q_e}{r^3}$$

$$= -k\sum_{mrs}(\delta_{ir}\delta_{ms}-\delta_{is}\delta_{mr})\,q_r\left[p_m\frac{q_s}{r^3} + p_s\frac{q_m}{r^3}\right] =$$

$$= -k\left(2q_i\frac{\bar{p}\cdot\bar{q}}{r^3} - \frac{\bar{p}\cdot\bar{q}\,q_i}{r^3} - p_i\frac{\bar{q}^2}{r^3}\right) = \qquad \bar{q}^2 = r^2$$

$$= -k q_i\frac{\bar{p}\cdot\bar{q}}{r^3} + k\frac{p_i}{r}$$

$$\{f, p^2\} = p\{f,p\} + \{f,p\}p$$

$$-\frac{k}{2}\left\{\frac{q_i}{r}, \sum_j p_j^2\right\} = -\frac{k}{2}\sum_j 2p_j\left\{\frac{q_i}{r}, p_j\right\} =$$

$$= +\frac{k}{2}\sum_j 2p_j\underbrace{\left\{p_j, \frac{q_i}{r}\right\}} =$$

$$-\frac{\partial}{\partial q_j}\frac{q_i}{r} = -\frac{\partial q_i}{\partial q_j}\cdot\frac{1}{r} - q_i\frac{\partial}{\partial q_j}\frac{1}{r} =$$

$$= -\delta_{ij} \frac{1}{r} + q_i \frac{q_j}{r^3}$$

$$= k \sum_j p_j \left(-\delta_{ij} \frac{1}{r} \right) + k \sum_j p_j \, q_i \frac{q_j}{r^3}$$

$$= -k \frac{p_i}{r} + k \, \bar{p} \cdot \bar{q} \, \frac{q_i}{r^3}$$

$$= 0 \quad //$$