LESSON 15.

1. The dimension of an intersection.

Our aim in this Lesson is to prove the following theorem on the dimension of the intersection of two algebraic varieties.

Theorem 1.1. *Let K be an algebraically closed field. Let $X, Y \subset \mathbb{P}^n$ be quasi–projective varieties. Assume that $X \cap Y \neq \emptyset$. Then if Z is any irreducible component of $X \cap Y$, then $\dim Z \geq \dim X + \dim Y - n$.*

To prove Theorem 1.1, the main ingredient will be the following theorem, known as "Krull's principal ideal theorem".

Theorem 1.2. *Let R be a noetherian ring, let $a \in R$ be a non-invertible element. Then, for any prime ideal $\mathcal{P} \subset R$, minimal over the ideal (a) generated by a, the height of $\mathcal{P}$ is at most 1, i.e. $ht\mathcal{P} \leq 1$. If moreover a is a non-zero divisor, then $ht\mathcal{P} = 1$.*

We postpone the proof of Theorem 1.2 to the end of this lesson and proceed to the proof of Theorem 1.1. It will be divided in three steps. Note first that, possibly passing to the closure, we can assume that X, Y are projective varieties. Then we can assume that $X \cap Y$ intersects $U_0 \simeq \mathbb{A}^n$, so, possibly after restricting X and Y to $\mathbb{A}^n$, we may work with irreducible closed subsets of the affine space. Put $r = \dim X$, $s = \dim Y$.

Step 1. Assume that $X = V(F)$ is an irreducible hypersurface, with F irreducible polynomial of $K[x_1, \ldots, x_n]$. The irreducible components of $X \cap Y$ correspond, by the Nullstellensatz, to the minimal prime ideals containing $I(X \cap Y)$ in $K[x_1, \ldots, x_n]$. We recall (Cor. 1.12, Lesson 4) that $I(X \cap Y) = \sqrt{I(X) + I(Y)} = \sqrt{\langle I(Y), F\rangle}$. So those prime ideals are the minimal prime ideals over $\langle I(Y), F\rangle$. They correspond bijectively to the minimal prime ideals containing $\langle f\rangle$ in $\mathcal{O}(Y)$, where f is the regular function on Y defined by F. We distinguish two cases:

(i) if $Y \subset X = V(F)$, then $f = 0$ and $Y \cap X = Y$; since $s = \dim Y > r + s - n = (n-1) + s - n$, the theorem is easily true in this case;

(ii) if $Y \not\subset X$, then $f \neq 0$, moreover f is not invertible, otherwise $X \cap Y = \emptyset$: hence the minimal prime ideals over $\langle f\rangle$ in $\mathcal{O}(Y)$, which is an integral domain, have all height one by Theorem 1.2. So for all Z, irreducible component of $X \cap Y$, $\dim Z = \dim Y - 1 = r + s - n$ (Theorem 1.8, Lesson 8).

Step 2. Assume that $I(X)$ is generated by $n-r$ polynomials (where $n-r$ is the codimension of X): $I(X) = \langle F_1, \dots, F_{n-r}\rangle$. Then we can argue by induction on $n-r$: we first intersect Y with $V(F_1)$, whose irreducible components are all hypersurfaces, and apply Step 1: all irreducible components of $Y \cap V(F_1)$ have dimension either s or $s-1$. Then we intersect each of these components with $V(F_2)$, and so on. We conclude that every irreducible component Z has $\dim Z \geq \dim Y - (n-r) = r+s-n$.

Step 3. We use the isomorphism $\psi : X \cap Y \simeq (X \times Y) \cap \Delta_{\mathbb{A}^n}$ (see Ex.1, Lesson 11). Note that $X \times Y$ is irreducible by Proposition 1.13, Lesson 7. ψ preserves the irreducible components and their dimensions, so we consider instead of X and Y, the algebraic sets $X \times Y$ and $\Delta_{\mathbb{A}^n}$, contained in $\mathbb{A}^{2n}$. We have $\dim X \times Y = r+s$ (Proposition 1.11, Lesson 8). $\Delta_{\mathbb{A}^n}$ is a linear subspace of $\mathbb{A}^{2n}$, so it satisfies the assumption of Step 2; indeed it has dimension n in $\mathbb{A}^{2n}$ and is defined by n linear equations. Hence, for all Z we have: $\dim Z \geq (r+s)+n-2n = r+s-n$. □

The above theorem can be seen as a generalization of the Grassmann relation for linear subspaces. However, it is not an existence theorem, because it says nothing about $X \cap Y$ being non–empty. But for projective varieties, the following more precise version of the theorem holds:

Theorem 1.3. *Let $X, Y \subset \mathbb{P}^n$ be projective varieties of dimensions r, s. If $r+s-n \geq 0$, then $X \cap Y \neq \emptyset$.*

Proof. Let $C(X)$, $C(Y)$ be the affine cones associated to X and Y. Then $C(X) \cap C(Y)$ is certainly non–empty, because it contains the origin $O(0, 0, \dots, 0)$. Assume we know that $C(X)$ has dimension $r+1$ and $C(Y)$ has dimension $s+1$: then by Theorem 1.1 all irreducible components Z of $C(X) \cap C(Y)$ have dimension $\geq (r+1)+(s+1)-(n+1) = r+s-n+1 \geq 1$, hence Z contains points different from O. These points give rise to points of $\mathbb{P}^n$ belonging to $X \cap Y$. The conclusion of the proof will follow from next proposition. □

Proposition 1.4. *Let $Y \subset \mathbb{P}^n$ be a projective variety.*

Then $\dim Y = \dim C(Y) - 1$. *If $S(Y)$ denotes the homogeneous coordinate ring, hence also* $\dim Y = \dim S(Y) - 1$.

Proof. Let $p : \mathbb{A}^{n+1} \setminus \{O\} \to \mathbb{P}^n$ be the canonical morphism. Let us recall that $C(Y) = p^{-1}(Y) \cup \{O\}$. Assume that $Y_0 := Y \cap U_0 \neq \emptyset$ and consider also $C(Y_0) = p^{-1}(Y_0) \cup \{O\}$. Then we have:

$$C(Y_0) = \{(\lambda, \lambda a_1, \dots, \lambda a_n) \mid \lambda \in K, (a_1, \dots, a_n) \in Y_0\}.$$

So we can define a birational map between $C(Y_0)$ and $Y_0 \times \mathbb{A}^1$ as follows:

$$(y_0, y_1, \dots, y_n) \in C(Y_0) \to ((y_1/y_0, \dots, y_n/y_0), y_0) \in Y_0 \times \mathbb{A}^1,$$

$$((a_1, \ldots, a_n), \lambda) \in Y_0 \times \mathbb{A}^1 \to (\lambda, \lambda a_1, \ldots, \lambda a_n) \in C(Y_0).$$

Therefore $\dim C(Y_0) = \dim(Y_0 \times \mathbb{A}^1) = \dim Y_0 + 1$. To conclude, it is enough to remark that $\dim Y = \dim Y_0$ and $\dim C(Y) = \dim C(Y_0) = \dim S(Y)$. □

We observe that also $C(Y)$ and $Y \times \mathbb{P}^1$ are birationally equivalent.

Corollary 1.5. *1. If $X, Y \subset \mathbb{P}^2$ are projective curves over an algebraically closed field, then $X \cap Y \neq \emptyset$.*
2. $\mathbb{P}^1 \times \mathbb{P}^1$ is not isomorphic to $\mathbb{P}^2$.

Proof. 1. is a straightforward application of Theorem 1.3.

To prove 2., assume by contradiction that $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ is an isomorphism. Let L, L' be two skew lines in $\mathbb{P}^1 \times \mathbb{P}^1$; since φ is an isomorphism, then $\varphi(L)$, $\varphi(L')$ are rational disjoint curves in $\mathbb{P}^2$, but this contradicts 1. □

If $X, Y \subset \mathbb{P}^n$ are varieties of dimensions r, s, then $r+s-n$ is called the *expected dimension* of $X \cap Y$. If all irreducible components Z of $X \cap Y$ have the expected dimension, then we say that the intersection $X \cap Y$ is *proper* or that *X and Y intersect properly.*

For example, two plane projective curves X, Y intersect properly if they don't have any common irreducible component. In this case, it is possible to predict the number of points of intersections. Precisely, it is possible to associate to every point $P \in X \cap Y$ a number $i(P; X, Y)$, called the *multiplicity of intersection of X and Y at P*, in such a way that

$$\sum_{P \in X \cap Y} i(P; X, Y) = dd',$$

where d is the degree of X and d' is the degree of Y. This result is the Theorem of Bézout, and is the first result of the branch of algebraic geometry called Intersection Theory. For a proof of the Theorem of Bézout, see for instance the classical [Walker, Algebraic curves], or [Fulton, Algebraic Curves].

Let X be a closed subvariety of $\mathbb{P}^n$ (resp. of $\mathbb{A}^n$) of codimension r. X is called a *complete intersection* if $I_h(X)$ (resp. $I(X)$) is generated by r polynomials, the minimum possible number.

Hence, if X is a complete intersection of codimension r, then X is certainly the intersection of r hypersurfaces. Conversely, if X is intersection of r hypersurfaces, then, by Theorem 1.1, using induction, we deduce that $\dim X \geq n - r$; even assuming equality, we cannot conclude that X is a complete intersection, but simply that $I(X)$ is the radical of an ideal generated by r polynomials.

Example 1.6. *The skew cubic (again).*

Let $X \subset \mathbb{P}^3$ be the skew cubic. The homogeneous ideal of X is generated by the three polynomials F_1, F_2, F_3, the 2×2–minors of the matrix

$$M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix},$$

which are linearly independent polynomials of degree 2. Note that $I_h(X)$ does not contain any linear polynomial, because X is not contained in any hyperplane, and that the homogeneous component of minimal degree 2 of $I_h(X)$ is a vector space of dimension 3. Hence $I_h(X)$ cannot be generated by two polynomials, i.e. X is not a complete intersection.

Nevertheless, X is the intersection of the surfaces $V_P(F)$, $V_P(G)$, where

$$F = F_1 = \begin{vmatrix} x_0 & x_1 \\ x_1 & x_2 \end{vmatrix} \text{ and } G = \begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_0 \end{vmatrix}.$$

Indeed, clearly $F, G \in I_h(X)$ so $X \subset V_P(F) \cap V_P(G)$. Conversely, observe that $G = x_0F - x_3(x_0x_3 - x_1x_2) + x_2(x_1x_3 - x_2^2)$. If $P[x_0, \dots, x_3] \in V_P(F) \cap V_P(G)$, then P is a zero of $x_0x_3^2 - 2x_1x_2x_3 + x_2^3$, and therefore also of

$$x_2(x_0x_3^2 - 2x_1x_2x_3 + x_2^3) = x_1^2x_3^2 - 2x_1x_2^2x_3 + x_2^4 = (x_1x_3 - x_2^2)^2 = F_3^2.$$

Hence P is a zero also of $F_3 = x_1x_3 - x_2^2$. So P annihilates $x_3(x_0x_3 - x_1x_2) = x_3F_2$. If P satisfies the equation $x_3 = 0$, then it satisfies also $x_2 = 0$ and $x_1 = 0$, therefore $P = [1, 0, 0, 0] \in X$. If $x_3 \neq 0$, then $P \in V_P(F_1, F_2, F_3) = X$.

The geometric description of this phenomenon is that the skew cubic X is the set-theoretic intersection of a quadric and a cubic, which are tangent along X, so their intersection is X "counted with multiplicity 2".

This example motivates the following definition: X is a *set–theoretic complete intersection* if codim$X = r$ and the ideal of X is the radical of an ideal generated by r polynomials. It is an open problem if all irreducible curves of $\mathbb{P}^3$ are set–theoretic complete intersections. For more details, see [Kunz].

We conclude this lesson with the proof of Krull's principal ideal Theorem 1.2.

Proof. Let $\mathcal{P}$ be a prime ideal, minimal among those containing (a), let $R_{\mathcal{P}}$ be the localization. Then $ht\mathcal{P} = \dim R_{\mathcal{P}}$, because of the bijection between prime ideals of $R_{\mathcal{P}}$ and prime ideals of R contained in $\mathcal{P}$. Moreover $\mathcal{P}R_{\mathcal{P}}$ is a minimal prime ideal over $aR_{\mathcal{P}}$, the ideal generated by a in $R_{\mathcal{P}}$. So, we can replace the ring R with its localization $R_{\mathcal{P}}$, or, in other words, we can assume that R is a local ring and that its maximal ideal $\mathcal{M}$ is minimal over (a).

It is enough to prove that, for any prime ideal $\mathcal{Q}$ of R, with $\mathcal{Q} \neq \mathcal{M}$, we have $ht\mathcal{Q} = 0$. Indeed this will imply $ht\mathcal{M} \leq 1$. Let $j : R \to R_{\mathcal{Q}}$ be the natural homomorphism. For any integer i, $i \geq 1$, we consider $\mathcal{Q}^i$, and its saturation with $\mathcal{Q}$: $\mathcal{Q}^{(i)} := j^{-1}(\mathcal{Q}^i R_{\mathcal{Q}})$, called the i-th symbolic power of $\mathcal{Q}$. It is $\mathcal{Q}$-primary. We have $\mathcal{Q}^i \subset \mathcal{Q}^{(i)}$ and

$$\mathcal{Q} = \mathcal{Q}^{(1)} \supseteq \mathcal{Q}^{(2)} \supseteq \cdots \supseteq \mathcal{Q}^{(i)} \supseteq \dots .$$

We also have

$$(a) + \mathcal{Q} \supseteq (a) + \mathcal{Q}^{(2)} \supseteq \cdots \supseteq (a) + \mathcal{Q}^{(i)} \supseteq \dots . \tag{1}$$

We observe that in $R/(a)$ there is only one prime ideal, $\mathcal{M}/(a)$, because R is local and $\mathcal{M}$ is minimal over (a), therefore $R/(a)$ has dimension 0; since it is noetherian of dimension 0, $R/(a)$ is artinian, and we can conclude that the chain of ideals (1) is stationary, so there exists an integer n such that $(a) + \mathcal{Q}^{(n)} = (a) + \mathcal{Q}^{(n+1)}$.

Let $q \in \mathcal{Q}^{(n)}$: so $q \in (a) + \mathcal{Q}^{(n+1)}$, and it can be written in the form $q = ra + q'$, where $r \in R$, $q' \in \mathcal{Q}^{(n+1)} \subset \mathcal{Q}^{(n)}$. Therefore $ra = q - q' \in \mathcal{Q}^{(n)}$; but $a \notin \mathcal{Q}$ (because $\mathcal{M}$ is minimal over (a)), and $\mathcal{Q}^{(n)}$ is $\mathcal{Q}$-primary, so $r \in \mathcal{Q}^{(n)}$. We conclude that $\mathcal{Q}^{(n)} = a\mathcal{Q}^{(n)} + \mathcal{Q}^{(n+1)}$.

We can apply now Nakayama's lemma (Theorem 1.7 below), and get $\mathcal{Q}^{(n)} = \mathcal{Q}^{(n+1)}$. Therefore $\mathcal{Q}^n R_{\mathcal{Q}} = \mathcal{Q}^{n+1} R_{\mathcal{Q}}$. We apply Nakayama's lemma again, and we conclude that $\mathcal{Q}^n R_{\mathcal{Q}} = (0)$. So every element of the maximal ideal $\mathcal{Q}R_{\mathcal{Q}}$ of $R_{\mathcal{Q}}$ is nilpotent, which implies that $ht\mathcal{Q}R_{\mathcal{Q}} = 0$.

We recall here the statement of Nakayama's lemma.

Theorem 1.7. *Let $I \subset R$ be an ideal contained in the Jacobson radical of R (the intersection of the maximal ideals). Let M be a finitely generated R-module, let $N \subset M$ be a submodule. If $M = N + IM$, then $M = N$.*

We have applied Nakayama's lemma the first time in the situation where R is a local ring and $I = (a) \subset \mathcal{M}$, which is the Jacobson radical of R. The R-module M is $\mathcal{Q}^{(n)}$ and its submodule N is $\mathcal{Q}^{(n+1)}$. The second time, we are instead in the situation where the ring is $R_{\mathcal{Q}}$, $I = \mathcal{Q}R_{\mathcal{Q}}$, the module M is $\mathcal{Q}^n R_{\mathcal{Q}}$ and N is (0).

To conclude the proof of the theorem, we observe that the second assertion follows from the first one, because if $\mathcal{P}$ is a prime ideal of height zero, all its elements are zero-divisors. Indeed, let $r \in \mathcal{P}$, $r \neq 0$; we can find an element $t \notin \mathcal{P}$ belonging to the intersection $\cap_i \mathcal{P}_i$ of the prime ideals of height zero different from $\mathcal{P}$ (there is a finite number of such ideals because R is noetherian). Otherwise $\mathcal{P} \subset \cap_i \mathcal{P}_i$, but this would imply $\mathcal{P} \subset \mathcal{P}_i$ for some i. Now observe that rt belongs to the intersection of all minimal prime ideals of R, so rt is nilpotent: there exists $\alpha \geq 0$ such that $(rt)^\alpha = 0$. Since $t \notin \mathcal{P}$, it is not nilpotent, so $t^\alpha \neq 0$.

Hence there is a minimum $\beta \geq 0$ such that $r^\beta t^\alpha \neq 0$ but $r^{\beta+1} t^\alpha = r(r^\beta t^\alpha) = 0$. This proves that r is a zero-divisor. $\square$

Exercises 1.8. 1. Let $X \subset \mathbb{P}^2$ be the union of three points not lying on a line. Prove that the homogeneous ideal of X cannot be generated by two polynomials.