homework 1 solutions

Exercise 1

Y is a random variable taking values in $\{1, ..., N\}$ and for each value $P(Y = j) = \frac{1}{N}$.

$$E[Y] = \sum_{j=1}^{N} j\frac{1}{N} = \frac{1}{N}\sum_{j=1}^{N} j = \frac{1}{N}\frac{N(N+1)}{2} = \frac{N+1}{2}$$

Exercise 2

From the definition of univariate normal distribution:

$$p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

where $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}_0$.
From the definition of expectation of continuous random variable:

$$\begin{aligned} E[X] &= \int_{-\infty}^{+\infty} \frac{x}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx. \end{aligned}$$

Let $t := \frac{x-\mu}{\sqrt{2\sigma^2}}$, hence $dt = \frac{dx}{\sqrt{2\sigma^2}}$. Then:

$$\begin{aligned} E[X] &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} \left(\sqrt{2\sigma^2}t + \mu\right) e^{-t^2} \sqrt{2\sigma^2}\, dt \\ &= \frac{\sqrt{2\sigma^2}}{\sqrt{2\pi\sigma^2}} \left[\int_{-\infty}^{+\infty} \left(\sqrt{2\sigma^2}t\right) e^{-t^2}\, dt + \int_{-\infty}^{+\infty} \mu e^{-t^2}\, dt\right]. \end{aligned}$$

Applying the Fundamental Theorem of Calculus we can conclude the proof,

$$\begin{aligned} E[X] &= \frac{1}{\sqrt{\pi}} \left[\sqrt{2\sigma^2}\left[-\frac{1}{2}e^{-t^2}\right]_{-\infty}^{+\infty} + \mu\sqrt{\pi}\right] \\ &= \frac{1}{\sqrt{\pi}}\left[0 + \mu\sqrt{\pi}\right] = \mu. \end{aligned}$$

Exercise 3

X and Y have discrete joint distribution:

$$p(x,y) = \begin{cases} \frac{1}{30}(x+y) & \text{for } x = 0,1,2 \text{ and } y = 0,1,2,3 \\ 0 & \text{otherwise} \end{cases}.$$

calculating X and Y marginal distributions we obtain

$$p(x) = \sum_{y=0}^{3} \frac{1}{30}(x+y) = \frac{1}{30}\left[\sum_{y=0}^{3} x + \sum_{y=0}^{3} y\right] = \frac{1}{15}(2x+3)$$

$$p(y) = \sum_{x=0}^{2} \frac{1}{30}(x+y) = \frac{1}{30}\left[\sum_{x=0}^{2} x + \sum_{x=0}^{2} y\right] = \frac{1}{10}(y+1)$$

multiplying them together we have

$$p(x)p(y) = \frac{1}{150}(2x+3)(y+1) = \frac{1}{150}(2xy+2x+3y+3)$$

which is different from p(x,y)

Exercise 4

From the definition of marginal density for continuous random variables:

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{+\infty} 24xy\, dx \\ &= \int_{-\infty}^{0} 24xy\, dx + \int_{0}^{1-y} 24xy\, dx + \int_{1-y}^{+\infty} 24xy\, dx \\ &= 0 + \int_{0}^{1-y} 24xy\, dx + 0 \\ &= y\int_{0}^{1-y} 24x\, dx. \end{aligned}$$

Applying the Fundamental Theorem of Calculus we can conclude that:

$$f_Y(y) = \begin{cases} 12y(1-y)^2 & \text{for } y \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$

From the definition of conditional density:

$$f_{X|Y=\frac{1}{2}}(x) = \begin{cases} \frac{f_{XY}(x,\frac{1}{2})}{f_Y(\frac{1}{2})} & \text{for } x \in (0,1),\ x+\frac{1}{2} < 1 \\ 0 & \text{otherwise} \end{cases}$$

From the solution (1) we can conclude that:

$$f_{X|Y=\frac{1}{2}}(x) = \begin{cases} 8x & \text{for } x \in (0,\frac{1}{2}) \\ 0 & \text{otherwise} \end{cases}$$

Exercise 5

The joint distribution of X and Y is:

$$f(x,y) = \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} x^{\alpha_1-1} y^{\alpha_2-1} (1-x-y)^{\alpha_3-1}$$

with $0 < x < 1, 0 < y < 1, x + y < 1$.

$$f(x) = \int_0^{1-x} f(x,y)dy$$

$$f(x) = \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} x^{\alpha_1-1} \int_0^{1-x} y^{\alpha_2-1}(1-x-y)^{\alpha_3-1}dy$$

performing the change of variables y=(1-x)t we obtain:

$$f(x) = \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} x^{\alpha_1-1}(1-x)^{\alpha_2+\alpha_3-1} \int_0^1 t^{\alpha_2-1}(1-t)^{\alpha_3-1}dt$$

now since the integral is a Beta function not normalized we have

$$\int_0^1 t^{\alpha_2-1}(1-t)^{\alpha_3-1}dt = \frac{\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(\alpha_2+\alpha_3)}$$

therefore we prove

$$f(x) = \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2+\alpha_3)} x^{\alpha_1-1}(1-x)^{\alpha_2+\alpha_3-1} = \text{Beta}(\alpha_1, \alpha_2+\alpha_3)$$