homework 2 solutions

Exercise 1

$$C = \{S, I\}$$

the variable corona (C) can take the value healthy (S) or sick (I)

$$T = \{P, N\}$$

the variable tampone (T) can take the value positive (P) or negative(N).

we know that

$$P(C = S) = 1 - P(C = I) = 0.995$$

hence

$$P(T = P) = P(C = I) * P(T = P|C = I) + P(C = S) * P(T = P|C = S) = 0.005 * 0.97 + 0.995 * 0.03 = 0.0347$$

with the bayes' theorem we have

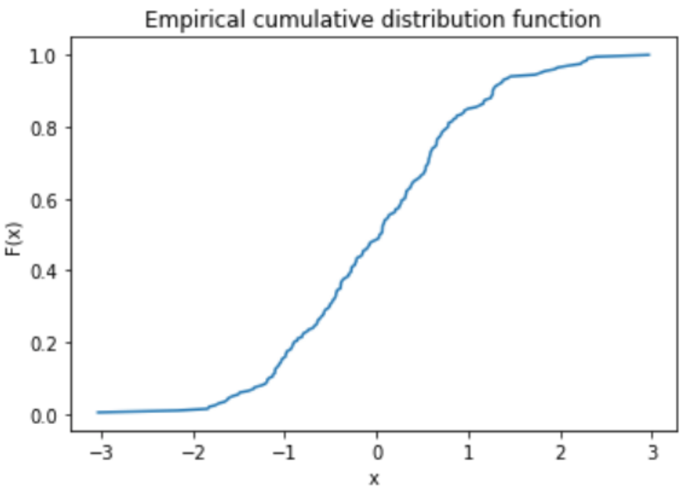
$$P(C = I|T = P) = \frac{P(T = P|C = I)P(C = I)}{P(T = P)} = \frac{0.98 * 0.005}{0.0347} = 0.1412$$

Exercise 2

First solution

```
def cdf(dist,x):
    sample=np.array([pyro.sample("n",dist) for i in range(200)]) #I obtain a sample
 of 200 elements from distribution dist
    sample_ord=np.sort(sample) #I sort the sample
    f=np.array([((i+1)/200) for i in range(200)]) #value of the empirical cumulative
function for each sorted element
    plt.plot(sample_ord,f)
    plt.title("Empirical cumulative distribution function")
    plt.xlabel("x")
    plt.ylabel("F(x)")
    plt.rcParams["figure.figsize"]=(15,5)
    a=sample_ord[sample_ord<=x]
    fdx=(len(a)/200) #value of cdf in x
    print("The value of cdf in " + str(x)+" is: F("+str(x)+") ="+str(fdx))
    return fdx

cdf(pyro.distributions.Normal(0.,1.),2)
```

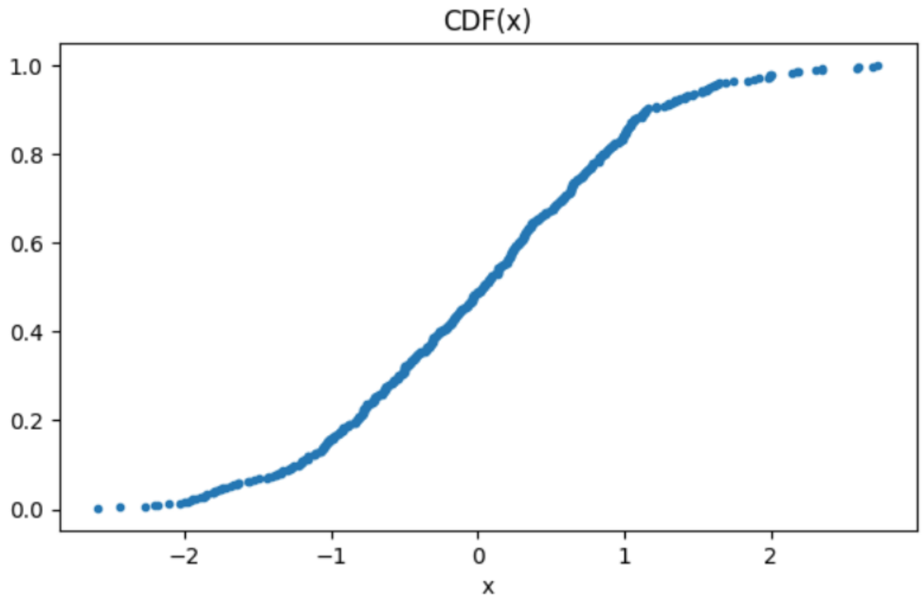
```
The value of cdf in 2 is: F(2) =0.965
```



Second solution

```
# Cumulative distribution function
def cdf(dist, val, iter):
    samples = [pyro.sample("n", dist) for i in range(iter)]
    u = np.sort(samples)
    t = list(filter(lambda i: i < val, u))
    n = len(u)
    v = np.arange(1, n + 1) / n
    z = []
    for i in range(len(t)):
        z.append(v[i])
    return t, z


# mean and std deviation
mu, sigma = 0, 1

# generate random data from a Normal distribution
rand_normal = dists.Normal(mu, sigma)

# max value for the cumulative distributive function
xMax = 1

# number of samples
samps = 500

# apply cdf
x, y = cdf(rand_normal, xMax, samps)
```

```
# plot
fig = plt.figure(figsize=(7,4))
_ = plt.plot(x, y, '.')
_ = plt.title("CDF(x)")
_ = plt.xlabel("x")
plt.show()
```



Exercise 3

3.1

If σ^2 has approximately 95% probability to be in the range $[22, 41]$ and it is distributed as an inverse-gamma $IG(\alpha = 38, \beta = 1110)$ we can write:

$$P(\sigma^2 \in [22, 41]) = 0.95 = \int_{22}^{41} f_{IG}(\sigma^2; 38, 110) d\sigma^2 = F_{IG}(\sigma^2 = 41; 38, 110) - F_{IG}(\sigma^2 = 22; 38, 110)$$

So we can use the function *ecdf* implemented in the previous exercise and compute the difference of the cumulative functions:

```
pyro.set_rng_seed(1)
dist2 = pyro.distributions.InverseGamma(38,1110)
a = 22
b = 41
print("Prior probability of sigma^2 in [22,41] is", ecdf(dist2,b)-ecdf(dist2,a))
```

```
Prior probability of sigma^2 in [22,41] is 0.954
```

3.2

The density function of an inverse gamma distribution is:

$$f(\sigma^2;\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}(\frac{1}{\sigma^2})^{\alpha+1}e^{-\frac{\beta}{\sigma^2}}.$$

The likelihood of the normal distibution with unknown variance is:

$$L(x;\sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2}(\frac{x_i-\mu}{\sigma})^2} = (\frac{1}{\sqrt{2\pi\sigma^2}})^n e^{-\frac{1}{2\sigma^2}\sum_{1=1}^{n}(x_i-\mu)^2}.$$

And the posterior distribution is given by:

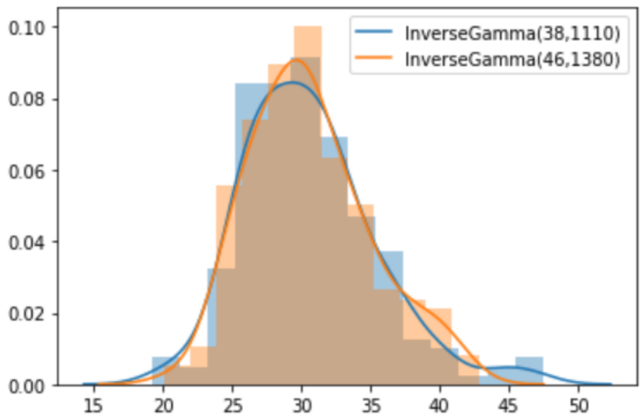
$$p(\theta|x) \propto p(x|\theta)p(\theta) \propto (\sigma^2)^{-\frac{n}{2}}e^{-\frac{1}{2\sigma^2}\sum_{1=1}^{n}(x_i-\mu)^2} * (\frac{1}{\sigma^2})^{\alpha+1}e^{-\frac{\beta}{\sigma^2}} = (\sigma^2)^{-(\frac{n}{2}+\alpha+1)}e^{-\frac{1}{\sigma^2}(\frac{\sum_{1=1}^{n}(x_i-\mu)^2}{2}+\beta)}$$

which looks like an

$$IG(\alpha+\frac{n}{2},\beta+\frac{1}{2}\sum_{i=1}^{N}(x-\mu)^2)$$

```
n=16
newalpha=38+n/2
data=[183, 173, 181, 170, 176, 180, 187, 176, 171, 190, 184, 173, 176, 179, 181, 186]
new=[]
for i in range(len(data)):
    new.append(pow(data[i]-180,2))
newbeta=1110+sum(new)*0.5

vardist=distr.InverseGamma(38,1110)
newvardist=distr.InverseGamma(newalpha,newbeta)
vardist_samples = [pyro.sample("prior",vardist) for i in range(200)]
newvardist_samples = [pyro.sample("post",newvardist) for i in range(200)]

sns.distplot(vardist_samples,kde_kws={"label": "InverseGamma(38,1110)"})
sns.distplot(newvardist_samples,kde_kws={"label": "InverseGamma(46,1380)"})
```

```
<matplotlib.axes._subplots.AxesSubplot at 0x7fe8f5b54750>
```



3.3
First solution

1. The posterior of the variance is:

$$P(\sigma^2|x) \propto e^{-\frac{1}{\sigma^2}\left[\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{2}+\beta\right]}\left(\frac{1}{\sigma^2}\right)^{\alpha+\frac{n}{2}+1}$$

Let $y = \sigma^2$ and $z = g(y) = \sqrt{y} = \sigma$. Thus, $g^{-1}(z) = z^2$ and $\frac{dg^{-1}(z)}{dz} = 2z$.
With this transformation the posterior of the variance becomes:

$$P(y|x) \propto e^{-\frac{1}{y}\left[\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{2}+\beta\right]}\left(\frac{1}{y}\right)^{\alpha+\frac{n}{2}+1}$$

and using the change of variable formula, we obtain:

$$P(z|x) = P((g^{-1}(z)|x)\left|\frac{dg^{-1}(z)}{dz}\right|$$

$$\propto |2z| e^{-\frac{1}{z^2}\left[\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{2}+\beta\right]}\left(\frac{1}{z^2}\right)^{\alpha+\frac{n}{2}+1}$$

Second solution

We know that

$$\int_0^\infty p_X(x)\,dx = 1$$

Let's substitute $y = \sqrt{x} \implies x = y^2$ and so $dx = 2y\,dy$

$$\int_0^\infty p_X(y^2)2y\,dy = 1$$

This means that, if we have $X = \sigma^2$ and $Y = \sigma$, we have the pdf for σ as:

$$p_\sigma(y) = 2yp_{\sigma^2}(y^2)$$

where the posteriror probability for σ^2 is an $InverseGamma(\sigma^2|\alpha + \frac{n}{2}, \beta + \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2})$

Therefore, the posterior probability of σ is:

$$p_\sigma(y) = 2y\frac{\beta^{\alpha+\frac{n}{2}}}{\Gamma(\alpha+\frac{n}{2})}\frac{e^{-\frac{2\beta+\sum_{i=1}^{n}(x_i-\mu)^2}{2y^2}}}{y^{2(\alpha+\frac{n}{2}+1)}}$$

Exercise 4

First solution

Let (Ω, F, P) a probability space and $(\mathbf{X}, A)$ a measurable space. Let $X, Y : (\Omega, F, P) \rightarrow (\mathbf{X}, A)$ two random variables s.t.

$$X \sim Exponential(\lambda) \Rightarrow p(x_i|\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i}$$
$$Y \sim Gamma(\alpha, \beta) \Rightarrow p(\lambda|\alpha, \beta) = \frac{\beta^{\alpha} \lambda^{\alpha-1} e^{-\beta\lambda}}{\Gamma(\alpha)}$$

where x_i is a generic component of $x = (x_1, \ldots, x_n)$.

From the definition of conjugate distribution, we want to prove that the posterior distribution is a Gamma distribution. For this reason, we can consider only the terms depending from λ:

$$p(\lambda|\alpha, \beta) \propto \lambda^{\alpha-1} e^{-\beta\lambda}$$

and:

$$p(\lambda|x) \propto p(x|\lambda)p(\lambda).$$

Then:

$$p(\lambda|x) \propto \left[\prod_{i=1}^{n} \lambda e^{-\lambda x_i}\right] \left[\lambda^{\alpha-1} e^{-\beta\lambda}\right] =$$
$$= \lambda^{n+\alpha-1} e^{-\lambda(\beta + n\bar{x})}$$

where: $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x.$

So $\lambda \sim Gamma(\alpha + n, \beta + n\bar{x})$, which proves our thesis.

Second solution

Given

$$X \sim \text{Exp}(x|\lambda) = \lambda e^{-\lambda x}$$
$$p(\lambda) = \text{Gamma}(\lambda|\alpha, \beta) = \frac{\lambda^{\alpha-1} \beta^{\alpha} e^{-\lambda\beta}}{\Gamma(\alpha)}$$

the posterior probability takes the form

$$\begin{aligned} p(\lambda|x, \alpha, \beta) &= \frac{\text{Exp}(x|\lambda)p(\lambda)}{p(x)} \\ &= \frac{\lambda e^{-\lambda x} \lambda^{\alpha-1} \beta^{\alpha} e^{-\lambda\beta}}{p(x)\Gamma(\alpha)} \\ &= \frac{\lambda^{\alpha} \beta^{\alpha} e^{-\lambda(\beta+x)}}{p(x)\Gamma(\alpha)} \\ &= \frac{\lambda^{(\alpha+1)-1} (\beta + x)^{(\alpha+1)} e^{-\lambda(\beta+x)}}{\Gamma(\alpha + 1)} \frac{\beta^{\alpha}\Gamma(\alpha + 1)}{(\beta + x)^{(\alpha+1)}\Gamma(\alpha)p(x)} \propto \text{Gamma}(\lambda|\alpha + 1, \beta + x) \end{aligned}$$