LESSON 17.

1. The tangent space and the notion of smoothness.

In this Lesson we follow the approach of Šafarevič. We define the tangent space $T_{X,P}$ at a point P of an *affine* variety $X \subset \mathbb{A}^n$ as the union of the lines passing through P and "touching" X at P. It results to be an affine subspace of $\mathbb{A}^n$. Then we will find a "local" characterization of $T_{X,P}$, this time interpreted as a vector space, the direction of $T_{X,P}$, only depending on the local ring $\mathcal{O}_{X,P}$: this will allow to define the tangent space at a point of any quasi–projective variety.

Assume first that $X \subset \mathbb{A}^n$ is closed and $P = O = (0, \dots, 0)$. Let L be a line through P: if $A(a_1, \dots, a_n)$ is another point of L, then a general point of L has coordinates $(ta_1, \dots, ta_n)$, $t \in K$. If $I(X) = (F_1, \dots, F_m)$, then the intersection $X \cap L$ is determined by the following system of equations in the indeterminate t:

$$F_1(ta_1, \dots, ta_n) = \cdots = F_m(ta_1, \dots, ta_n) = 0.$$

The solutions of this system of equations are the roots of the greatest common divisor $G(t)$ of the polynomials $F_1(ta_1, \dots, ta_n), \dots, F_m(ta_1, \dots, ta_n)$ in $K[t]$, i.e. the generator of the ideal they generate. We may factorize $G(t)$ as $G(t) = ct^e(t - \alpha_1)^{e_1} \dots (t - \alpha_s)^{e_s}$, where $c \in K$, $\alpha_1, \dots, \alpha_s \neq 0$, $e, e_1, \dots, e_s$ are non-negative, and $e > 0$ if and only if $P \in X \cap L$. The number e is by definition the **intersection multiplicity at P of X and L.** If $G(t)$ is identically zero, then $L \subset X$ and the intersection multiplicity is, by definition, $+\infty$.

Note that the polynomial $G(t)$ doesn't depend on the choice of the generators $F_1, \dots, F_m$ of $I(X)$, but only on the ideal $I(X)$ and on L.

Definition 1.1. The line L is **tangent to the variety X at P** if the intersection multiplicity of L and X at P is at least 2 (in particular, if $L \subset X$). The **tangent space to X at P** is the union of the lines that are tangent to X at P; it is denoted $T_{P,X}$.

We will see now that $T_{P,X}$ is an affine subspace of $\mathbb{A}^n$. Assume that $P \in X$: then the polynomials F_i may be written in the form $F_i = L_i + G_i$, where L_i is a homogeneous linear polynomial (possibly zero) and G_i contains only terms of degree ≥ 2. Then

$$F_i(ta_1, \dots, ta_n) = tL_i(a_1, \dots, a_n) + G_i(ta_1, \dots, ta_n),$$

where the last term is divisible by t^2. Let L be the line $\overline{OA}$, with $A = (a_1, \dots, a_n)$. We note that the intersection multiplicity of X and L at P is the maximal power of t dividing the greatest common divisor, so L is tangent to X at P if and only if $L_i(a_1, \dots, a_n) = 0$ for all $i = 1, \dots, m$.

Therefore the point A belongs to $T_{P,X}$ if and only if

$$L_1(a_1, \dots, a_n) = \dots = L_m(a_1, \dots, a_n) = 0.$$

This shows that $T_{P,X}$ is a linear subspace of $\mathbb{A}^n$, whose equations are the linear components of the equations defining X.

Example 1.2. *(i)* $T_{O,\mathbb{A}^n} = \mathbb{A}^n$*, because* $I(\mathbb{A}^n) = (0)$.

(ii) If X *is a hypersurface, with* $I(X) = (F)$*, we write as above* $F = L + G$*; then* $T_{O,X} = V(L)$*: so* $T_{O,X}$ *is either a hyperplane if* $L \neq 0$*, or the whole space* $\mathbb{A}^n$ *if* $L = 0$*. For instance, if* X *is the affine plane cuspidal cubic* $V(x^3 - y^2) \subset \mathbb{A}^2$, $T_{O,X} = \mathbb{A}^2$.

Assume now that $P \in X$ has coordinates $(y_1, \dots, y_n)$. With a linear transformation we may translate P to the origin $(0, \dots, 0)$, taking as new coordinates functions on $\mathbb{A}^n$ $x_1 - y_1, \dots, x_n - y_n$. This corresponds to considering the K-isomorphism $K[x_1, \dots, x_n] \longrightarrow K[x_1 - y_1, \dots, x_n - y_n]$, which takes a polynomial $F(x_1, \dots, x_n)$ to its Taylor expansion

$$G(x_1 - y_1, \dots, x_n - y_n) = F(y_1, \dots, y_n) + d_P F + d_P^{(2)} F + \dots,$$

where $d_P^{(i)} F$ denotes the i^{th} differential of F at P: it is a homogeneous polynomial of degree i in the variables $x_1 - y_1, \dots, x_n - y_n$. In particular the linear term is

$$d_P F = \frac{\partial F}{\partial x_1}(P)(x_1 - y_1) + \dots + \frac{\partial F}{\partial x_n}(P)(x_n - y_n).$$

We get that, if $I(X) = (F_1, \dots, F_m)$, then $T_{P,X}$ is the affine subspace of $\mathbb{A}^n$ defined by the equations

$$d_P F_1 = \dots = d_P F_m = 0.$$

The affine space $\mathbb{A}^n$, which may identified with K^n, can be given a natural structure of K-vector space with origin P, so in a natural way $T_{P,X}$ is a vector subspace (with origin P). The functions $x_1 - y_1, \dots, x_n - y_n$ form a basis of the dual space $(K^n)^*$ and their restrictions generate $T^*_{P,X}$. Note moreover that $\dim T^*_{P,X} = k$ if and only if $n - k$ is the maximal number of polynomials linearly independent among $d_P F_1, \dots, d_P F_m$. If $d_P F_1, \dots, d_P F_{n-k}$ are these polynomials, then they form a base of the orthogonal $T^\perp_{P,X}$ of the vector space $T_{P,X}$ in $(K^n)^*$, because they vanish on $T_{P,X}$.

Let us define now the *differential of a regular function.* Let $f \in \mathcal{O}(X)$ be a regular function on X. We want to define the differential of f at P. Since X is closed in $\mathbb{A}^n$, f is induced by a polynomial $F \in K[x_1, \dots, x_n]$ as well as by all polynomials of the form $F + G$ with $G \in I(X)$. Fix $P \in X$: then $d_P(F + G) = d_P F + d_P G$ so the differentials of two polynomials inducing the same function f on X differ by the term $d_P G$ with $G \in I(X)$. By definition, $d_P G$ is zero along $T_{P,X}$, so we may define $d_p f$ as a regular function on $T_{P,X}$, the differential of f at P: it is the function on $T_{P,X}$ induced by $d_P F$. Since $d_P F$ is a linear combination of $x_1 - y_1, \ldots, x_n - y_n$, $d_p f$ can also be seen as an element of $T^*_{P,X}$.

There is a natural map $d_p : \mathcal{O}(X) \to T^*_{P,X}$, which sends f to $d_p f$. Because of the rules of derivation, it is clear that $d_P(f+g) = d_P f + d_P g$ and $d_P(fg) = f(P)d_P g + g(P)d_P f$. In particular, if $c \in K$, $d_p(cf) = c d_P f$. So d_p is a linear map of K-vector spaces. We denote again by d_P the restriction of d_P to $I_X(P)$, the maximal ideal of the regular functions on X which are zero at P. Since clearly $f = f(P) + (f - f(P))$ then $d_P f = d_P(f - f(P))$, so this restriction doesn't modify the image of the map.

Proposition 1.3. *The map $d_P : I_X(P) \to T^*_{P,X}$ is surjective and its kernel is $I_X(P)^2$. Therefore $T^*_{P,X} \simeq I_X(P)/I_X(P)^2$ as K-vector spaces.*

Proof. Let $\varphi \in T^*_{P,X}$ be a linear form on $T_{P,X}$. φ is the restriction of a linear form on K^n: $\lambda_1(x_1 - y_1) + \ldots + \lambda_n(x_n - y_n)$, with $\lambda_1, \ldots, \lambda_n \in K$. Let G be the polynomial of degree 1 $\lambda_1(x_1 - y_1) + \ldots + \lambda_n(x_n - y_n)$: the function g induced by G on X is zero at P and coincides with its own differential, so d_p is surjective.

Let now $g \in I_X(P)$ such that $d_p g = 0$, g induced by a polynomial G. Note that $d_P G$ may be interpreted as a linear form on K^n which vanishes on $T_{P,X}$, hence as an element of $T^\perp_{P,X}$. So $d_P G = c_1 d_p F_1 + \ldots + c_m d_p F_m$ ($c_1, \ldots, c_m$ suitable elements of K). Let us consider the polynomial $G - c_1F_1 - \ldots - c_mF_m$: since its differential at P is zero, it doesn't have any term of degree 0 or 1 in $x_1 - y_1, \ldots, x_n - y_n$, so it belongs to $I(P)^2$. Since $G - c_1F_1 - \ldots - c_mF_m$ defines the function g on X, we conclude that $g \in I_X(P)^2$. □

Corollary 1.4. *The tangent space $T_{P,X}$ is isomorphic to $(I_X(P)/I_X(P)^2)^*$ as an abstract K-vector space.*

Corollary 1.5. *Let $\varphi : X \to Y$ be an isomorphism of affine varieties and $P \in X$, $Q = \varphi(P)$. Then the tangent spaces $T_{P,X}$ and $T_{Q,Y}$ are isomorphic.*

Proof. φ induces the comorphism $\varphi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$, which results to be an isomorphism such that $\varphi^* I_Y(Q) = I_X(P)$ and $\varphi^* I_Y(Q)^2 = I_X(P)^2$. So there is an induced homomorphism

$$I_Y(Q)/I_Y(Q)^2 \to I_X(P)/I_X(P)^2.$$

which is an isomorphism of K-vector spaces. By dualizing we get the claim. □

The above map from $T_{P,X}$ to $T_{Q,Y}$ is called the *differential of φ at P* and is denoted by $d_P\varphi$.

Now we would like to find a "more local" characterization of $T_{P,X}$. To this end we consider the local ring of P in X: $\mathcal{O}_{P,X}$. We recall the natural map $\mathcal{O}(X) \to \mathcal{O}_{P,X} = \mathcal{O}(X)_{I_X(P)}$, the last one being the localization. It is natural to extend the map $d_P : \mathcal{O}(X) \to T^*_{P,X}$ to $\mathcal{O}_{P,X}$ setting

$$d_P\Big(\frac{f}{g}\Big) = \frac{g(P)d_P f - f(P)d_P g}{g(P)^2}.$$

As in the proof of Proposition 1.3 one proves that the map $d_P : \mathcal{O}_{P,X} \to T^*_{P,X}$ induces an isomorphism $\mathcal{M}_{P,X}/\mathcal{M}^2_{P,X} \to T^*_{P,X}$, where $\mathcal{M}_{P,X}$ is the maximal ideal of $\mathcal{O}_{P,X}$. So by duality we have: $T_{P,X} \simeq (\mathcal{M}_{P,X}/\mathcal{M}^2_{P,X})^*$. This proves that the tangent space $T_{P,X}$ is a *local invariant* of P in X.

Definition 1.6. Let X be any quasi-projective variety, $P \in X$. The *Zariski tangent space* of X at P is the vector space $(\mathcal{M}_{P,X}/\mathcal{M}^2_{P,X})^*$.

It is an abstract vector space, but if $X \subset \mathbb{A}^n$ is closed, taking the dual of the comorphism associated to the inclusion morphism $X \hookrightarrow \mathbb{A}^n$, we have an embedding of $T_{P,X}$ into $T_{P,\mathbb{A}^n} = \mathbb{A}^n$. If $X \subset \mathbb{P}^n$ and $P \in U_i = \mathbb{A}^n$, then $T_{P,X} \subset U_i$: its projective closure $\mathbb{T}_{P,X}$ is called the *embedded tangent space* to X at P.

As we have seen the tangent space $T_{P,X}$ is invariant by isomorphism. In particular its dimension is invariant. If $X \subset \mathbb{A}^n$ is closed, $I(X) = (F_1, \dots, F_m)$, then $\dim T_{P,X} = n - r$, where r is the dimension of the K-vector space generated by $\{d_P F_1, \dots, d_p F_m\}$.

Since $d_P F_i = \frac{\partial F_i}{\partial x_1}(P)(x_1 - y_1) + \dots + \frac{\partial F_i}{\partial x_n}(P)(x_n - y_n)$, r is the rank of the following $m \times n$ matrix, the *Jacobian matrix of* X *at* P:

$$J(P) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(P) & \dots & \frac{\partial F_1}{\partial x_n}(P) \\ \dots & \dots & \dots \\ \frac{\partial F_m}{\partial x_1}(P) & \dots & \frac{\partial F_m}{\partial x_n}(P) \end{pmatrix}.$$

The *generic Jacobian matrix of* X is instead the following matrix with entries in $\mathcal{O}(X)$:

$$J = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \dots & \frac{\partial F_1}{\partial x_n} \\ \dots & \dots & \dots \\ \frac{\partial F_m}{\partial x_1} & \dots & \frac{\partial F_m}{\partial x_n} \end{pmatrix}.$$

The rank of J is ρ when all minors of order $\rho + 1$ are functions identically zero on X, while at least one minor of order ρ is different from zero at some point. Hence, for all $P \in X$ rk $J(P) \le \rho$, and rk $J(P) < \rho$ if and only if all minors of order ρ of J vanish at P. It is then clear that there is a non-empty open subset of X where $\dim T_{P,X}$ is minimal, equal to $n-\rho$, and a proper (possibly empty) closed subset formed by the points P such that $\dim T_{P,X} > n-\rho$.

Definition 1.7. The points of an irreducible variety X for which $\dim T_{P,X} = n-\rho$ (the minimal) are called *smooth* or *non-singular* (or *simple) points* of X. The remaining points are called *singular* (or multiple). X is a *smooth* variety if all its points are smooth.

If X is quasi-projective, the same argument may be repeated for any affine open subset.

Example 1.8. *Let $X \subset \mathbb{A}^n$ be the irreducible hypersurface $V(F)$. Then $J = (\frac{\partial F}{\partial x_1} \dots \frac{\partial F}{\partial x_n})$ is a row matrix. So rk $J = 0$ or 1. If rk $J = 0$, then $\frac{\partial F}{\partial x_i} = 0$ in $\mathcal{O}(X)$ for all i. So $\frac{\partial F}{\partial x_i} \in I(Y) = (F)$. Since the degree of $\frac{\partial F}{\partial x_i}$ is $\leq \deg F - 1$, it follows that $\frac{\partial F}{\partial x_i} = 0$ in the polynomial ring. If the characteristic of K is zero this means that F is constant: a contradiction. If char $K = p$, then $F \in K[x_1^p, \dots, x_n^p]$; since K is algebraically closed, then all coefficients of F are p-th powers, so $F = G^p$ for a suitable polynomial G; but again this is impossible because F is irreducible. So always rk $J = 1 = \rho$. Hence for P general in X, i.e. for P varying in a suitable non-empty open subset of X, $\dim T_{P,X} = n-1$. For some particular points, the singular points of X, we can have $\dim T_{P,X} = n$, i.e. $T_{P,X} = \mathbb{A}^n$.*

So in the case of a hypersurface $\dim T_{P,X} \geq \dim X$ for every point P in X, and equality holds in the points of the smooth locus of X. The general case can be reduced to the case of hypersurfaces in view of the following theorem.

Theorem 1.9. *Every quasi-projective irreducible variety X is birational to a hypersurface in some affine space.*

Proof. We observe that we can reduce the proof to the case in which X is affine, closed in $\mathbb{A}^n$. Let $m = \dim X$. We have to prove that the field of rational functions $K(X)$ is isomorphic to a field of the form $K(t_1, \dots, t_{m+1})$, where $t_1, \dots, t_{m+1}$ satisfy only one non-trivial relation $F(t_1, \dots, t_{m+1}) = 0$, where F is an irreducible polynomial with coefficients in K. This will follow from the "Abel's primitive element Theorem" concerning extensions of fields. To state it, we need some preliminaries.

Let $K \subset L$ be an extension of fields. Let $a \in L$ be algebraic over K, and let $f_a \in K[x]$ be its minimal polynomial: it is irreducible and monic. Let E be the splitting field of f_a.

Definition 1.10. An element a, algebraic over K, is *separable* if f_a does not have any multiple root in E, i.e. if f_a and its derivative f'_a don't have any common factor of positive degree. Otherwise a is inseparable. If $K \subset L$ is an algebraic extension of fields, it is called separable if any element of L is separable.

In view of the fact that f_a is irreducible, and that the GCD of two polynomials is independent of the field where one considers the coefficients, if a is inseparable, then f'_a is the zero polynomial. If char $K = 0$, this implies that f_a is constant, which is a contradiction. So in characteristic 0, any algebraic extension is separable. If char $K = p > 0$, then $f_a \in K[x^p]$, and f_a is called an inseparable polynomial. In particular algebraic inseparable elements can exist only in positive characteristic. On the other hand, if the characteristic of K is $p > 0$ and K is algebraically closed, if $f_a = a_0 + a_1x^p + a_2x^{2p} + \cdots + a_kx^{kp}$, then all coefficients are p-th powers in K, i.e. $a_i = b_i^p$ for suitable elements b_i; therefore $f_a = b_0^p + b_1^px^p + b_2^px^{2p} + \cdots + b_k^px^{kp} = (b_0 + b_1x + b_2x^2 + \cdots + b_kx^k)^p$, and this contradicts the irreducibility of f_a.

Theorem 1.11. *[Abel's primitive element Theorem.]*

Let $K \subseteq L = K(y_1, \ldots, y_m)$ be an algebraic finite extension. If L is a separable extension, then there exists $\alpha \in L$, called a primitive element of L, such that $L = K(\alpha)$ is a simple extension.

We can now prove Theorem 1.9. The field of rational functions of X is of the form $K(X) = Q(K[X]) = K(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are the coordinate functions on X and $tr.d.K(X)/K = m$. Possibly after renumbering them, we can assume that the first m coordinate functions $t_1, \ldots, t_m$ are algebraically independent over K, and $K(X)$ is an algebraic extension of $L := K(t_1, \ldots, t_m)$. So in our situation we can apply Theorem 1.11: there exists a primitive element α such that $K(X) = L(\alpha) = K(t_1 \ldots, t_m, \alpha)$. So there exists an irreducible polynomial $f \in L[x]$ such that $K(X) = L[x]/(f)$. Multiplying f by a suitable element of $K[t_1, \ldots, t_m]$, invertible in L, we can eliminate the denominator of f and replace f by a polynomial $g \in K[t_1, \ldots, t_m, x] \subset L[x]$. Now $K[t_1, \ldots, t_m, x]/(g)$ is contained in $L[x](g) = K(X)$, and its quotient field is again $K(X)$. But $K[t_1, \ldots, t_m, x]/(g)$ is the coordinate ring of the hypersurface $Y \subset \mathbb{A}^{m+1}$ of equation $g = 0$. It is clear that X and Y are birationally equivalent, because they have the same fiekd of rational functions. This concludes the proof.

One can show that the coordinate functions on Y, $t_1, \ldots, t_{m+1}$, can be chosen to be linear combinations of the original coordinate functions on X: this means that Y is obtained as a suitable birational projection of X.

Theorem 1.12. *The dimension of the tangent space at a non-singular point of an irreducible variety X is equal to* $\dim X$.

Proof. It is enough to prove the claim under the assumption that X is affine. Let Y be an affine hypersurface birational to X (which exists by the previous theorem) and $\varphi : X \dashrightarrow Y$ be a birational map. There exist open non-empty subsets $U \subset X$ and $V \subset Y$ such that $\varphi : U \to Y$ is an isomorphism. The set of smooth points of Y is an open subset W of Y

such that $W \cap V$ is non-empty and $\dim T_{P,Y} = \dim Y = \dim X$ for all $P \in W \cap V$. But $\varphi^{-1}(W \cap V) \subset U$ is open non-empty and $\dim T_{Q,X} = \dim X$ for all $Q \in \varphi^{-1}(W \cap V)$. This proves the theorem. □

Now we would like to study a variety X in a neighbourhood of a smooth point. We have seen that P is smooth for X if and only if $\dim T_{P,X} = \dim X$. Assume X affine: in this case the local ring of P in X is $\mathcal{O}_{P,X} \simeq \mathcal{O}(X)_{I_X(P)}$. But by Theorem 1.8, Lesson 8, we have: $\dim \mathcal{O}_{P,X} = \mathrm{ht}\mathcal{M}_{P,X} = \mathrm{ht}I_X(P) = \dim \mathcal{O}(X) = \dim X$ and $\dim T_{P,X} = \dim_K \mathcal{M}_{P,X}/\mathcal{M}^2_{P,X}$. Therefore P is smooth if and only if

$$\dim_K \mathcal{M}_{P,X}/\mathcal{M}^2_{P,X} = \dim \mathcal{O}_{P,X}$$

(the first one is a dimension as K-vector space, the second one is a Krull dimension). By the Nakayama's Lemma a basis of $\mathcal{M}_{P,X}/\mathcal{M}^2_{P,X}$ corresponds bijectively to a minimal system of generators of the ideal $\mathcal{M}_{P,X}$ (observe that the residue field of $\mathcal{O}_{P,X}$ is K). Therefore P is smooth for X if and only if $\mathcal{M}_{P,X}$ is minimally generated by r elements, where $r = \dim X$, in other words if and only if $\mathcal{O}_{P,X}$ is a *regular local ring.*

For example, if X is a curve, P is smooth if and only if $T_{P,X}$ has dimension 1, i.e. $\mathcal{M}_{P,X}$ is principal: $\mathcal{M}_{P,X} = (t)$. This means that the equation $t = 0$ only defines the point P, i.e. P has one local equation in a suitable neighborhood of P.

Let P be a smooth point of X and $\dim X = n$. Functions $u_1, \dots, u_n \in \mathcal{O}_{P,X}$ are called *local parameters* at P if $u_1, \dots, u_n \in \mathcal{M}_{P,X}$ and their residues $\bar{u}_1, \dots, \bar{u}_n$ in $\mathcal{M}_{P,X}/\mathcal{M}^2_{P,X}$ $(= T^*_{P,X})$ form a basis, or equivalently if $u_1, \dots, u_n$ is a minimal set of generators of $\mathcal{M}_{P,X}$. Recalling the isomorphism

$$d_P : \mathcal{M}_{P,X}/\mathcal{M}^2_{P,X} \to T^*_{P,X}$$

we deduce that $u_1, \dots, u_n$ are local parameters if and only if $d_P\bar{u}_1, \dots, d_P\bar{u}_n$ are linearly independent forms on $T_{P,X}$ (which is a vector space of dimension n), if and only if the system of equations on $T_{P,X}$

$$d_P\bar{u}_1 = \ldots = d_P\bar{u}_n = 0$$

has only the trivial solution P (which is the origin of the vector space $T_{P,X}$.

Let $u_1, \dots, u_n$ be local parameters at P. There exists an open affine neighborhood of P on which $u_1, \dots, u_n$ are all regular. We replace X by this neighborhood, so we assume that X is affine and that $u_1, \dots, u_n$ are polynomial functions on X. Let X_i be the closed subset $V(u_i)$ of X: it has codimension 1 in X, because u_i is not identically zero on X ($u_1, \dots, u_n$ is a minimal set of generators of $\mathcal{M}_{P,X}$).

Proposition 1.13. *In this notation, P is a smooth point of X_i, for all $i = 1, \dots, n$, and $\bigcap_i T_{P,X_i} = \{P\}$.*

Proof. Assume that U_i is a polynomial inducing u_i, then $X_i = V(U_i) \cap X = V(I(X) + (U_i))$. So $I(X_i) \supset I(X) + (U_i)$. By considering the linear parts of the polynomials of the previous ideal, we get: $T_{P,X_i} \subset T_{P,X} \cap V(d_P U_i)$. By the assumption on the u_i, it follows that $T_{P,X} \cap V(d_P U_1) \cap \dots \cap V(d_P U_n) = \{P\}$. Since $\dim T_{P,X} = n$, we can deduce that $T_{P,X} \cap V(d_P U_i)$ is strictly contained in $T_{P,X}$, and $\dim T_{P,X} \cap V(d_P U_i) = n - 1$. So $\dim T_{P,X_i} \leq n - 1 = \dim X_i$, hence P is a smooth point on X_i, equality holds and $T_{P,X_i} = T_{P,X} \cap V(d_P U_i)$. Moreover $\bigcap T_{P,X_i} = \{P\}$. □

Note that $\bigcap_i X_i$ has no positive-dimensional component Y passing through P: otherwise the tangent space to Y at P would be contained in T_{P,X_i} for all i, against the fact that $\bigcap T_{P,X_i} = \{P\}$.

Definition 1.14. Let X be a smooth variety. Subvarieties $Y_1, \dots, Y_r$ of X are called *transversal at P*, with $P \in \bigcap Y_i$, if the intersection of the tangent spaces T_{P,Y_i} has dimension as small as possible, i.e. if $\operatorname{codim}_{T_{P,X}}(\bigcap T_{P,Y_i}) = \sum \operatorname{codim}_X Y_i$.

Taking $T_{P,X}$ as ambient variety, one gets the relation:

$$\dim \bigcap T_{P,Y_i} \geq \sum \dim T_{P,Y_i} - (r-1) \dim T_{P,X};$$

hence

$$\operatorname{codim}_{T_{P,X}}(\bigcap T_{P,Y_i}) = \dim T_{P,X} - \dim \bigcap T_{P,Y_i} \leq \sum (\dim T_{P,X} - \dim T_{P,Y_i}) =$$
$$= \sum \operatorname{codim}_{T_{P,X}}(T_{P,Y_i}) \leq \sum \operatorname{codim}_X Y_i.$$

If equality holds, P is a smooth point for Y_i for all i, moreover we get that P is a smooth point for the set $\bigcap Y_i$.

For example, if X is a surface and $P \in X$ is smooth, there is a neighbourhood U of P such that P is the transversal intersection of two curves in U, corresponding to local parameters u_1, u_2. If P is singular we need three functions u_1, u_2, u_3 to generate the maximal ideal $\mathcal{M}_{P,X}$.