LESSON 16.

1. Complete varieties.

We work over an algebraically closed field K.

In this lesson, we will prove that the algebra of regular functions $\mathcal{O}(X)$ of an irreducible projective variety X is the base field K, i.e. that the only regular functions on X are the constants. We will obtain this theorem as a consequence of the theorem of completeness of projective varieties. The property of a variety to be complete can be seen as an analogue of compactness in the context of algebraic geometry.

Definition 1.1. Let X be a quasi–projective variety. X is *complete* if, for any quasi–projective variety Y, the natural projection on the second factor $p_2 : X \times Y \to Y$ is a closed map.

Note that both projections p_1, p_2 are morphisms: see Exercise 3, Lesson 14.

We recall that a topological space X is compact if and only if the above projection map is closed with respect to the product topology. Here the product variety $X \times Y$ does not carry the product topology but the Zariski topology, that is in general strictly finer (Proposition 1.2, lesson 3).

Example 1.2. *The affine line $\mathbb{A}^1$ is not complete: let $X = Y = \mathbb{A}^1$, $p_2 : \mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2 \to \mathbb{A}^1$ is the map such that $(x_1, x_2) \to x_2$. Then $Z := V(x_1x_2 - 1)$ is closed in $\mathbb{A}^2$ but $p_2(Z) = \mathbb{A}^1 \setminus \{O\}$ is not closed.*

Proposition 1.3. *(i) If $f : X \to Y$ is a regular map and X is complete, then $f(X)$ is a closed complete subvariety of Y.*

(ii) If X is complete, then all closed subvarieties of X are complete.

Proof. (i) Let $\Gamma_f \subset X \times Y$ be the graph of f: $\Gamma_f = \{(x, f(x)) \mid x \in X\}$. It is clear that $f(X) = p_2(\Gamma_f)$, so to prove that $f(X)$ is closed it is enough to check that Γ_f is closed in $X \times Y$. Let us consider the diagonal of Y: $\Delta_Y = \{(y, y) \mid y \in Y\} \subset Y \times Y$. If $Y \subset \mathbb{P}^n$, then $\Delta_Y = \Delta_{\mathbb{P}^n} \cap (Y \times Y)$, so it is closed in $Y \times Y$, because $\Delta_{\mathbb{P}^n}$ is the closed subset defined in $\Sigma_{n,n}$ by the equations $w_{ij} - w_{ji} = 0$, $i, j = 0, \dots, n$. There is a natural map $f \times 1_Y : X \times Y \to Y \times Y$, $(x, y) \to (f(x), y)$, such that $(f \times 1_Y)^{-1}(\Delta_Y) = \Gamma_f$. It is easy to see that $f \times 1_Y$ is regular, so Γ_f is closed, so also $f(X)$ is closed.

Let now Z be any variety and consider $p_2 : f(X) \times Z \to Z$ and the regular map $f \times 1_Z : X \times Z \to f(X) \times Z$. There is a commutative diagram:

$$\begin{array}{ccc} X \times Z & \xrightarrow{p_2'} & Z \\ \downarrow {\scriptstyle f\times 1_Z} & \nearrow & {\scriptstyle p_2} \\ f(X) \times Z & & \end{array}$$

If $T \subset f(X) \times Z$, then $(f \times 1_Z)^{-1}(T)$ is closed and $p_2(T) = p_2'((f \times 1_Z)^{-1}(T))$ is closed because X is complete. We conclude that $f(X)$ is complete.

(ii) Let $T \subset X$ be a closed subvariety and Y be any variety. We have to prove that $p_2 : T \times Y \to Y$ is closed. If $Z \subset T \times Y$ is closed, then Z is closed also in $X \times Y$, hence $p_2(Z)$ is closed because X is complete. □

Corollary 1.4. *1. If X is a complete variety, then $\mathcal{O}(X) \simeq K$.*

2. If X is an affine complete irreducible variety, then X is a point.

Proof. 1. If $f \in \mathcal{O}(X)$, f can be interpreted as a regular map $f : X \to \mathbb{A}^1$. By Proposition 1.3, (i), $f(X)$ is a closed complete subvariety of $\mathbb{A}^1$, which is not complete. Hence $f(X)$ has dimension < 1 and is irreducible, hence it is a point, so $f \in K$.

2. By part 1., $\mathcal{O}(X) \simeq K$. But $\mathcal{O}(X) \simeq K[x_1, \dots, x_n]/I(X)$, hence $I(X)$ is maximal. By the Nullstellensatz, X is a point. □

Before stating the Theorem 1.6 of completeness of projective varieties, we give a characterization of the closed subsets of a biprojective space $\mathbb{P}^n \times \mathbb{P}^m$, that will be needed in its proof. It is expressed in terms of equations in two series of variables, corresponding to the homogeneous coordinates $[x_0, \dots, x_n]$ on $\mathbb{P}^n$ and $[y_0, \dots, y_m]$ on $\mathbb{P}^m$.

Let $\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ be the Segre map. A closed subvariety X in $\mathbb{P}^N$ is defined by finitely many equations $F_k(w_{00}, \dots, w_{nm})$, where the F_k are homogeneous polynomials in the w_{ij}. On the subvariety $X \cap \Sigma$, where Σ is the Segre variety, we have $w_{ij} = x_i y_j$, so we can make this substitution in F_k and get equations $G_k(x_0, \dots, x_n; y_0, \dots, y_m) = 0$, where $G_k = F_k(x_0 y_0, \dots, x_n y_m)$: they are equations characterizing the subset $\sigma^{-1}(X)$. Note that each G_k is homogeneos in each set of variables x_i and y_j, and in the same degree in both.

Conversely, it is easy to see that a polynomial with this property of bihomogeneity can always be written as a polynomial in the products $x_i y_j$, and the possible ambiguity depending on the choice disappears in view of the equations of the Segre variety. So it describes a subset of $\mathbb{P}^n \times \mathbb{P}^m$ whose image in σ is closed. However, equations that are bihomogeneous in x_i and y_j always define an algebraic subvariety of $\mathbb{P}^n \times \mathbb{P}^m$ even if the degrees of homogeneity in the two sets of variables are different. Indeed if $G(x_0, \dots, x_n; y_0, \dots, y_m)$ has degree r in

x_i and s in y_j, and for instance $r > s$, then the equation $G = 0$ is equivalent to the system of equations $y_i^{r-s}G = 0$, $i = 0, \dots, m$, and these define an algebraic variety.

We will need the answer to the analogous question for the product $\mathbb{P}^n \times \mathbb{A}^m$. Let us assume that $\mathbb{A}^m = U_0 \subset \mathbb{P}^m$, defined by $y_0 \neq 0$. If we have a closed subset of $\mathbb{P}^n \times \mathbb{P}^m$ defined by equations $G_k(x_0, \dots, x_n; y_0, \dots, y_m) = 0$, with G_k homogeneous of degree r_k in y_j, dividing by $y_0^{r_k}$ and setting $v_j = y_j/y_0$, we get equations $g_k(x_0, \dots, x_n; v_1, \dots, v_m) = 0$ that are homogeneous in the x_i and in general non-homogeneous in the v_j.

These observations can be collected in the following result.

Theorem 1.5. *A subset $X \subset \mathbb{P}^n \times \mathbb{P}^m$ is a closed algebraic subvariety if and only if it is defined by a system of equations $G_k(x_0, \dots, x_n; y_0 \dots, y_m) = 0$, homogeneous separately in each set of variables. Every closed algebraic subvariety of $\mathbb{P}^n \times \mathbb{A}^m$ is defined by a system of equations $g_k(x_0, \dots, x_n; v_1, \dots, v_m) = 0$ that are homogeneous in $x_0, \dots, x_n$.*

Theorem 1.6. *Let $X \subset \mathbb{P}^n$ be a projective irreducible variety. Then X is complete.*

Proof. (see Šafarevič, Theorem 3, Ch.1, §5)

1. It is enough to prove that $p_2 : \mathbb{P}^n \times \mathbb{A}^m \to \mathbb{A}^m$ is closed, for any positive n, m. This can be observed by using the local character of closedness and the existence of an affine open covering of any quasi–projective varieties.

Indeed, let us assume first that $p_2 : \mathbb{P}^n \times Y \to Y$ is a closed map for any quasi-projective variety Y. We observe that $X \times Y$ is closed in $\mathbb{P}^n \times Y$, because X is closed in $\mathbb{P}^n$. So, if $Z \subset X \times Y$ is closed, it is also closed in $\mathbb{P}^n \times Y$, which implies that $p_2(Z)$ is closed in Y. So we can replace X with $\mathbb{P}^n$.

Secondly, since being closed is a local property, it is enough to cover Y by affine open subsets U_i, and prove the theorem for each of them. Hence we can assume that Y is an affine variety. Finally, if $Y \subset \mathbb{A}^m$ is closed, then $\mathbb{P}^n \times Y$ is closed in $\mathbb{P}^n \times \mathbb{A}^m$, so it is enough to prove theorem in the particular case $X = \mathbb{P}^n$ and $Y = \mathbb{A}^m$.

2. If $x_0, \dots, x_n$ are homogeneous coordinates on $\mathbb{P}^n$ and $y_1, \dots, y_m$ are non-homogeneous coordinates on $\mathbb{A}^m$, then any closed subvariety of $\mathbb{P}^n \times \mathbb{A}^m$ can be characterised as the set of common zeroes of a set of polynomials in the variables $x_0, \dots, x_n, y_1, \dots, y_m$, homogeneous in the first group of variables $x_0, \dots, x_n$ (Theorem 1.5).

3. Let $Z \subset \mathbb{P}^n \times \mathbb{A}^m$ be closed. Then Z is the set of solutions of a system of equations

$$\{G_i(x_0, \dots, x_n; y_1, \dots, y_m) = 0, i = 1, \dots, t,$$

where G_i is homogeneous in the x's. A point $P(\overline{y}_1, \dots, \overline{y}_m)$ is in $p_2(Z)$ if and only if the system

$$\{G_i(x_0, \dots, x_n; \overline{y}_0, \dots, \overline{y}_m) = 0, i = 1, \dots, t,$$

has a solution in $\mathbb{P}^n$, i.e. if the ideal of $K[x_0,\dots,x_n]$ generated by $G_1(x;\overline{y})$,..., $G_t(x;\overline{y})$ has at least one zero in $\mathbb{P}^n$. Hence

$$\begin{aligned} p_2(Z) &= \{(\overline{y}_1,\dots,\overline{y}_m)|\ \forall\ d\geq 1\ \langle G_1(x;\overline{y}),\dots,G_t(x;\overline{y})\rangle \not\supset K[x_0,\dots,x_n]_d\} \\ &= \bigcap_{d\geq 1}\{(\overline{y}_1,\dots,\overline{y}_m)|\ \langle G_1(x;\overline{y}),\dots,G_t(x;\overline{y})\rangle \not\supset K[x_0,\dots,x_n]_d\}. \end{aligned} \tag{1}$$

Let $\{M_\alpha\}_{\alpha=1,\dots,\binom{n+d}{d}}$ be the set of the monomials of degree d in $K[x_0,\dots,x_n]$; let $d_i = \deg\ G_i(x;\overline{y})$; let $\{N_i^\beta\}$ be the set of the monomials of degree $d-d_i$; let finally $T_d = \{(\overline{y}_1,\dots,\overline{y}_m)|\ \langle G_1(x;\overline{y}),\dots,G_t(x;\overline{y})\rangle \not\supset K[x_0,\dots,x_n]_d\}$. So equation (1) says that $p_2(Z) = \cap_{d\geq 1}T_d$, and to conclude the proof of the theorem it is enough to prove that T_d is closed in $\mathbb{P}^m$ for any $d\geq 1$.

Note that $P(\overline{y}_1,\dots,\overline{y}_m)\notin T_d$ if and only if $M_\alpha = \sum_i G_i(x;\overline{y})F_{i,\alpha}(x_0,\dots,x_n)$, for all α and for suitable polynomials $F_{i,\alpha}$ homogeneous of degree $d-d_i$. So $P\notin T_d$ if and only if, for all index α, M_α is a linear combination of the polynomials $\{G_i(x;\overline{y})N_i^\beta\}$, i.e. the matrix A of the coordinates of the polynomials $G_i(x;\overline{y})N_i^\beta$ with respect to the basis $\{M_\alpha\}$ has maximal rank $\binom{n+d}{d}$. So T_d is the set of zeroes of the minors of a fixed order of the matrix A, hence it is closed. □

Corollary 1.7. *Let X be a projective variety. Then $\mathcal{O}(X)\simeq K$.*

Corollary 1.8. *Let X be a projective variety, $\varphi: X\to Y\subset\mathbb{P}^n$ be any regular map. Then $\varphi(X)$ is a projective variety. In particular, if $X\simeq Y$, then Y is projective.*

Corollary 1.8 says that the notion of projective variety, differently from that of affine variety, is invariant by isomorphism, i.e. quasi-projective varieties that are isomorphic to projective varieties are already projective.

In algebraic terms, Theorem 1.6 can be seen as a result in Elimination Theory. Indeed it can be reformulated by saying that, given a system of algebraic equations in two sets of variables, $x_0,\dots,x_n$ and $y_1,\dots,y_m$, homogeneous in the first ones, it is possible to find another system of algebraic equations only in $y_1,\dots,y_m$, such that $\bar{y}_1,\dots,\bar{y}_m$ is a solution of the second system if and only if there exist $\bar{x}_0,\dots,\bar{x}_n$, that, together with $\bar{y}_1,\dots,\bar{y}_m$, are a solution of the first system. In other words, it is possible to eliminate a set of homogeneous variables from any system of algebraic equations.

Example 1.9. *Let $S=K[x_0,\dots,x_n]$. Let $d\geq 1$ be an integer number and consider S_d, the vector space of homogeneous polynomials of degree d. As an application of Theorem 1.6, we shall prove that the set of (proportionality classes of) reducible polynomials is a projective algebraic set in $\mathbb{P}(S_d)$.*

We denote by $X \subset \mathbb{P}(S_d)$ *the set of reducible polynomials. For any integer* k, $0 < k < d$, *let* $X_k \subseteq X$ *be the set of polynomials of the form* F_1F_2 *with* $\deg F_1 = k, \deg F_2 = d - k$. *Then* $X = \bigcup_{k=1}^{d-1} X_k$. *Let* $f_k : \mathbb{P}(S_k) \times \mathbb{P}(S_{d-k}) \to \mathbb{P}(S_d)$ *be the multiplication of polynomials, i.e.* $f_k([F_1], [F_2]) = [F_1F_2]$. f_k *is clearly a regular map, and its image is* $X_k = X_{d-k}$. *Since the domain is a projective variety, and precisely a Segre variety, it follows from Theorem 1.6 that* X_k *is also projective.*

In the special case d $= 2$, *the quadratic polynomials, the equations of* $X = X_1$ *are the minors of order* 3 *of the matrix associated to the quadric.*