LESSON 11.

1. Regular maps.

In this Lesson we will always assume that K is an algebraically closed field.

Let X, Y be quasi–projective varieties (or more generally locally closed sets). Let $\varphi : X \to Y$ be a map.

Definition 1.1. φ is a *regular map* or a *morphism* if

(i) φ is continuous for the Zariski topology;

(ii) φ preserves regular functions, i.e. for all $U \subset Y$ (U open and non–empty) and for all $f \in \mathcal{O}(U)$, then $f \circ \varphi \in \mathcal{O}(\varphi^{-1}(U))$:

$$\begin{array}{ccccc} X & \xrightarrow{\varphi} & Y & & \\ \uparrow & & \uparrow & & \\ \varphi^{-1}(U) & \xrightarrow{\varphi|} & U & \xrightarrow{f} & K \end{array}$$

Note that:

a) for all X the identity map $1_X : X \to X$ is regular;

b) for all X, Y, Z and regular maps $X \xrightarrow{\varphi} Y$, $Y \xrightarrow{\psi} Z$, the composite map $\psi \circ \varphi$ is regular.

An *isomorphism* of varieties is a regular map which possesses regular inverse, i.e. a regular map $\varphi : X \to Y$ such that there exists a regular map $\psi : Y \to X$ verifying the conditions $\psi \circ \varphi = 1_X$ and $\varphi \circ \psi = 1_Y$. In this case X and Y are said to be isomorphic, and we write: $X \simeq Y$.

If $\varphi : X \to Y$ is regular, there is a natural K–homomorphism $\varphi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$, called the *comorphism associated to* φ, defined by: $f \to \varphi^*(f) := f \circ \varphi$.

The construction of the comorphism is *functorial*, which means that:

a) $1_X^* = 1_{\mathcal{O}(X)}$;

b) $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.

This implies that, if $X \simeq Y$, then $\mathcal{O}(X) \simeq \mathcal{O}(Y)$. In fact, if $\varphi : X \to Y$ is an isomorphism and ψ is its inverse, then $\varphi \circ \psi = 1_Y$, so $(\varphi \circ \psi)^* = \psi^* \circ \varphi^* = (1_Y)^* = 1_{\mathcal{O}(Y)}$ and similarly $\psi \circ \varphi = 1_X$ implies $\varphi^* \circ \psi^* = 1_{\mathcal{O}(X)}$.

Example 1.2.

1) The homeomorphism $\varphi_i : U_i \to \mathbb{A}^n$ of Lesson 3, 1.6, is an isomorphism.

2) There exist homeomorphisms which are not isomorphisms. Let $Y = V(x^3 - y^2) \subset \mathbb{A}^2$. We have seen (see Exercise 2, Lesson 8) that $K[Y] \not\simeq K[\mathbb{A}^1]$, hence Y is not isomorphic to the affine line $\mathbb{A}^1$. Nevertheless, the map

$$\varphi : \mathbb{A}^1 \to Y \text{ such that } t \to (t^2, t^3)$$

is regular, bijective and also a homeomorphism (see Exercise 1, Lesson 8).

Its inverse $\varphi^{-1} : Y \to \mathbb{A}^1$ is defined by

$$(x, y) \to \begin{cases} \frac{y}{x} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad (x, y) = (0, 0). \end{cases}$$

Note that φ^{-1} cannot be regular at the point $(0, 0)$.

Next Proposition tells us how a morphism is given in practice, when the codomain is contained in an affine space.

Proposition 1.3. *Let $\varphi : X \to Y \subset \mathbb{A}^n$ be a map. Then φ is regular if and only if $\varphi_i := t_i \circ \varphi$ is a regular function on X, for all $i = 1, \dots, n$, where $t_1, \dots, t_n$ are the coordinate functions on Y.*

Proof. If φ is regular, then $\varphi_i = \varphi^*(t_i)$ is regular by definition.

Conversely, assume that φ_i is a regular function on X for all i. Let $Z \subset Y$ be a closed subset and we have to prove that $\varphi^{-1}(Z)$ is closed in X. Since any closed subset of $\mathbb{A}^n$ is an intersection of hypersurfaces, it is enough to consider $\varphi^{-1}(Y \cap V(F))$ with $F \in K[x_1, \dots, x_n]$:

$$\varphi^{-1}(Y \cap V(F)) = \{P \in X | F(\varphi(P)) = F(\varphi_1, \dots, \varphi_n)(P) = 0\} = V(F(\varphi_1, \dots, \varphi_n)).$$

But note that $F(\varphi_1, \dots, \varphi_n) \in \mathcal{O}(X)$: it is the composition of F with the regular functions $\varphi_1, \dots, \varphi_n$. Hence $\varphi^{-1}(Y \cap V(F))$ is closed, so we can conclude that φ is continuous. If $U \subset Y$ and $f \in \mathcal{O}(U)$, for any point P of U choose an open neighbourhood U_P such that $f = F_P/G_P$ on U_P. So $f \circ \varphi = F_P(\varphi_1, \dots, \varphi_n)/G_P(\varphi_1, \dots, \varphi_n)$ on $\varphi^{-1}(U_P)$, hence it is regular on each $\varphi^{-1}(U_P)$ and by consequence on $\varphi^{-1}(U)$. □

Remark. If $\varphi : X \to Y$ is a regular map and $Y \subset \mathbb{A}^n$, by Proposition 1.3 we can represent φ in the form $\varphi = (\varphi_1, \dots, \varphi_n)$, where $\varphi_1, \dots, \varphi_n \in \mathcal{O}(X)$ and $\varphi_i = \varphi^*(t_i)$. $\varphi_1, \dots, \varphi_n$ are not arbitrary in $\mathcal{O}(X)$ but such that Im $\varphi \subset Y$.

If Y is closed in $\mathbb{A}^n$, let us recall that $t_1, \dots, t_n$ generate $\mathcal{O}(Y)$, hence $\varphi_1, \dots, \varphi_n$ generate $\varphi^*(\mathcal{O}(Y))$ as K-algebra. This observation is the key for the following important result.

Theorem 1.4. *Let X be a locally closed algebraic set and Y be an affine algebraic set. Let $Hom(X,Y)$ denote the set of regular maps from X to Y and $Hom(\mathcal{O}(Y),\mathcal{O}(X))$ denote the set of $K-$ homomorphisms from $\mathcal{O}(Y)$ to $\mathcal{O}(X)$.*

Then the map $Hom(X,Y)\to Hom(\mathcal{O}(Y),\mathcal{O}(X))$, such that $\varphi : X\to Y$ goes to $\varphi^* : \mathcal{O}(Y)\to\mathcal{O}(X)$, is bijective.

Proof. Let $Y\subset\mathbb{A}^n$ and let $t_1,\dots,t_n$ be the coordinate functions on Y, so $\mathcal{O}(Y)=K[t_1,\dots,t_n]$. Let $u:\mathcal{O}(Y)\to\mathcal{O}(X)$ be a K–homomorphism: we want to define a morphism $u^\sharp : X\to Y$ whose associated comorphism is u. By the previous Remark, if $u^\sharp$ exists, its components have to be $u(t_1),\dots,u(t_n)$. So we define

$$\begin{array}{rccc} u^\sharp : & X & \to & \mathbb{A}^n \\ & P & \to & (u(t_1)(P)),\dots,u(t_n)(P)). \end{array}$$

This is a morphism by Proposition 1.3. We claim that $u^\sharp(X)\subset Y$. Let $F\in I(Y)$ and $P\in X$: then

$$\begin{aligned} F(u^\sharp(P)) &= F(u(t_1)(P),\dots,u(t_n)(P)) = \\ &= F(u(t_1),\dots,u(t_n))(P) = \\ &= u(F(t_1,\dots,t_n))(P) \text{ because } u \text{ is } K\text{-homomorphism} = \\ &= u(0)(P) = \\ &= 0(P) = 0. \qquad (1) \end{aligned}$$

So $u^\sharp$ is a regular map from X to Y.

We consider now $(u^\sharp)^* : \mathcal{O}(Y)\to\mathcal{O}(X)$: it takes a function f to $f\circ u^\sharp = f(u(t_1),\dots,u(t_n)) = u(f)$, so $(u^\sharp)^* = u$. Conversely, if $\varphi : X\to Y$ is regular, then $(\varphi^*)^\sharp$ takes P to

$$(\varphi^*(t_1)(P),\dots,\varphi^*(t_n)(P)) = (\varphi_1(P),\dots,\varphi_n(P)),$$

so $(\varphi^*)^\sharp = \varphi$. □

Note that, by definition, $1^\sharp_{\mathcal{O}(X)} = 1_X$, for all affine X; moreover $(v\circ u)^\sharp = u^\sharp\circ v^\sharp$ for all $u:\mathcal{O}(Z)\to\mathcal{O}(Y)$, $v:\mathcal{O}(Y)\to\mathcal{O}(X)$, K–homomorphisms of rings of regular functions of affine algebraic sets: this means that also this construction is functorial.

The construction of the comorphism associated to a regular function and the result of Theorem 1.4 can be rephrased using the language of categories. We will see it in Lesson 12.

If X and Y are quasi–projective varieties and $\varphi : X\to Y$ is a regular map, it is not always possible to extend the comorphism $\varphi^* : \mathcal{O}(Y)\to\mathcal{O}(X)$ to a homomorphism between the fields of rational functions $K(Y)\to K(X)$. Indeed, if f is a rational function on Y

with $\mathrm{dom} f = U$, it can happen that $\varphi(X) \cap \mathrm{dom} f = \emptyset$, in which case $f \circ \varphi$ does not exist. Nevertheless, if we assume that φ is **dominant**, i.e. $\overline{\varphi(X)} = Y$, then certainly $\varphi(X) \cap U \neq \emptyset$, hence $\langle \varphi^{-1}(U), f \circ \varphi \rangle \in K(X)$. We obtain a K–homomorphism, which is necessarily injective, $K(Y) \to K(X)$, also denoted by φ^*.

Note that in this case $K(X)$ contains the isomorphic image $\varphi^*(K(Y)) \simeq K(Y)$, therefore $tr.d.K(X)/K \geq tr.d.K(Y)/K$ and we have: $\dim X \geq \dim Y$. As above, it is possible to check that, if $X \simeq Y$, then $K(X) \simeq K(Y)$, hence $\dim X = \dim Y$. Moreover, if $P \in X$ and $Q = \varphi(P)$, then φ^* induces a map $\mathcal{O}_{Q,Y} \to \mathcal{O}_{P,X}$, such that $\varphi^* \mathcal{M}_{Q,Y} \subset \mathcal{M}_{P,X}$. This can be expressed by saying that $\varphi^* : \mathcal{O}_{Q,Y} \to \mathcal{O}_{P,X}$ is a local homomorphism. Also in this case, if φ is an isomorphism, then $\mathcal{O}_{Q,Y} \simeq \mathcal{O}_{P,X}$.

We will see now how to express in practice a regular map when the target is contained in a projective space. Let $X \subset \mathbb{P}^n$ be a quasi–projective variety and $\varphi : X \to \mathbb{P}^m$ be a map.

Proposition 1.5. *φ is a morphism if and only if, for any $P \in X$, there exist an open neighbourhood U_P of P and $n+1$ homogeneous polynomials $F_0, \dots, F_m$ of the same degree in $K[x_0, x_1, \dots, x_n]$, such that, if $Q \in U_P$, then $\varphi(Q) = [F_0(Q), \dots, F_m(Q)]$. In particular, for any $Q \in U_P$, there exists an index i such that $F_i(Q) \neq 0$.*

Proof. "$\Rightarrow$" Let $P \in X$, $Q = \varphi(P)$ and assume that $Q \in U_0$. Then $U := \varphi^{-1}(U_0)$ is an open neighbourhood of P and we can consider the restriction $\varphi|_U : U \to U_0$, which is regular. Possibly after restricting U, using non–homogeneous coordinates on U_0, we can assume that $\varphi|_U = (F_1/G_1, \dots, F_m/G_m)$, where (F_1, G_1), ..., (F_m, G_m) are pairs of homogeneous polynomials of the same degree such that $V_P(G_i) \cap U = \emptyset$ for all index i. We can reduce the fractions F_i/G_i to a common denominator F_0, so that $\deg F_0 = \deg F_1 = \cdots = \deg F_m$ and $\varphi|_U = (F_1/F_0, \dots, F_m/F_0) = [F_0, F_1, \dots, F_m]$, with $F_0(Q) \neq 0$ for $Q \in U$.

"$\Leftarrow$" Possibly after restricting U_P, we can assume $F_i(Q) \neq 0$ for all $Q \in U_P$ and suitable i. Let $i = 0$: then $\varphi|_{U_P} : U_P \to U_0$ operates as follows:

$$\varphi|_{U_P}(Q) = (F_1(Q)/F_0(Q), \dots, F_m(Q)/F_0(Q)),$$

so it is a morphism by Proposition 1.3. From this remark, one deduces that also φ is a morphism. $\square$

Example 1.6.

Let $X \subset \mathbb{P}^2$, $X = V_P(x_1^2 + x_2^2 - x_0^2)$, the projective closure of the unitary circle. We define $\varphi : X \to \mathbb{P}^1$ by

$$[x_0, x_1, x_2] \to \begin{cases} [x_0 - x_2, x_1] & \text{if } (x_0 - x_2, x_1) \neq (0,0) \\ [x_1, x_0 + x_2] & \text{if } (x_1, x_0 + x_2) \neq (0,0). \end{cases}$$

φ is well–defined because, on X, $x_1^2 = (x_0 - x_2)(x_0 + x_2)$. Moreover

$$(x_1, x_0 - x_2) \neq (0,0) \Leftrightarrow [x_0, x_1, x_2] \in X \setminus \{[1,0,1]\},$$

$$(x_0 + x_2, x_1) \neq (0,0) \Leftrightarrow [x_0, x_1, x_2] \in X \setminus \{[1,0,-1]\}.$$

The map φ is the natural extension of the rational function $f : X \setminus \{[1,0,1]\} \to K$ such that $[x_0, x_1, x_2] \to x_1/(x_0 - x_2)$ (Lesson 10, Example 1.11, 2). Now the point $P[1,0,1]$, the centre of the stereographic projection, goes to the point at infinity of the line $V_P(x_2)$.

By geometric reasons φ is invertible and $\varphi^{-1} : \mathbb{P}^1 \to X$ takes $[\lambda, \mu]$ to $[\lambda^2+\mu^2, 2\lambda\mu, \mu^2-\lambda^2]$ (note the connection with the Pitagorean triples!).

Indeed: the line through P and $[\lambda, \mu, 0]$ has equation: $\mu x_0 - \lambda x_1 - \mu x_2 = 0$. Its intersections with X are represented by the system:

$$\begin{cases} \mu x_0 - \lambda x_1 - \mu x_2 = 0 \\ x_1^2 + x_2^2 - x_0^2 = 0 \end{cases}$$

Assuming $\mu \neq 0$ this system is equivalent to the following:

$$\begin{cases} \mu x_0 - \lambda x_1 - \mu x_2 = 0 \\ \mu^2 x_0^2 = \mu^2(x_1^2 + x_2^2) = (\lambda x_1 + \mu x_2)^2 \end{cases}$$

Therefore, either $x_1 = 0$ and $x_0 = x_2$, or

$$\begin{cases} (\mu^2 - \lambda^2)x_1 - 2\lambda\mu x_2 = 0 \\ \mu x_0 = \lambda x_1 + \mu x_2 \end{cases}$$

which gives the required expression.

Example 1.7. *Affine transformations.*

Let $A = (a_{ij})$ be a $n \times n$ matrix with entries in K, let $B = (b_1, \dots, b_n) \in \mathbb{A}^n$ be a point. The map $\tau_A : \mathbb{A}^n \to \mathbb{A}^n$ defined by $(x_1, \dots, x_n) \to (y_1, \dots, y_n)$, such that

$$\{y_i = \sum_j a_{ij} x_j + b_i, i = 1, \dots, n,$$

is a regular map called an affine transformation of $\mathbb{A}^n$. In matrix notation τ_A is $Y = AX + B$. If A is of rank n, then τ_A is said non–degenerate and is an isomorphism: the inverse map τ_A^{-1} is represented by $X = A^{-1}Y - A^{-1}B$. More in general, an affine transformation from $\mathbb{A}^n$ to $\mathbb{A}^m$ is a map represented in matrix form by $Y = AX + B$, where A is a $m \times n$ matrix and $B \in \mathbb{A}^m$. It is injective if and only if $\mathrm{rk}A = n$ and surjective if and only if $\mathrm{rk}A = m$.

The isomorphisms of an algebraic set X in itself are called **automorphisms of** X: they form a group for the usual composition of maps, denoted by $Aut\ X$. If $X = \mathbb{A}^n$, the non–degenerate affine transformations form a subgroup of $Aut\ \mathbb{A}^n$.

If $n = 1$ and the characteristic of K is 0, then $Aut\ \mathbb{A}^1$ coincides with this subgroup. In fact, let $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ be an automorphism: it is represented by a polynomial $F(x)$ such that there exists $G(x)$ satisfying the condition $G(F(t)) = t$ for all $t \in \mathbb{A}^1$, i.e. $G(F(x)) = x$ in the polynomial ring $K[x]$. Then, taking derivatives, we get $G'(F(x))F'(x) = 1$, which implies $F'(t) \neq 0$ for all $t \in K$, so $F'(x)$ is a non–zero constant. Hence, F is linear and G is linear too.

If $n \geq 2$, then $Aut\ \mathbb{A}^n$ is not completely described. There exist non–linear automorphisms of degree d, for all d. For example, for $n = 2$: let $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$ be given by $(x, y) \to (x, y + P(x))$, where P is any polynomial of $K[x]$. Then $\varphi^{-1} : (x', y') \to (x', y' - P(x'))$. A very important and difficult open problem in Algebraic Geometry is the Jacobian conjecture, stating that, in characteristic zero, a regular map $\varphi : \mathbb{A}^n \to \mathbb{A}^n$ is an automorphism if and only if the Jacobian determinant $\mid J(\varphi) \mid$ is a non-zero constant.

Example 1.8. *Projective transformations.*

Let A be a $(n + 1) \times (n + 1)$–matrix with entries in K. Let $P[x_0, \dots, x_n] \in \mathbb{P}^n$: then $[a_{00}x_0 + \cdots + a_{0n}x_n, \dots, a_{n0}x_0 + \cdots + a_{nn}x_n]$ is a point of $\mathbb{P}^n$ if and only if it is different from $[0, \dots, 0]$. So A defines a regular map $\tau : \mathbb{P}^n \to \mathbb{P}^n$ if and only if $\mathrm{rk}A = n + 1$. If $\mathrm{rk}A = r < n + 1$, then A defines a regular map whose domain is the quasi–projective variety $\mathbb{P}^n \setminus \mathbb{P}(kerA)$. If $\mathrm{rk}A = n + 1$, then τ is an isomorphism, called a projective transformation. Note that the matrices λA, $\lambda \in K^*$, all define the same projective transformation. So $PGL(n + 1, K) := GL(n + 1, K)/K^*$ acts on $\mathbb{P}^n$ as the group of projective transformations.

If $X, Y \subset \mathbb{P}^n$, they are called **projectively equivalent** if there exists a projective transformation $\tau : \mathbb{P}^n \to \mathbb{P}^n$ such that $\tau(X) = Y$.

Theorem 1.9. *Fundamental theorem on projective transformations.*

Let two $(n + 2)$–tuples of points of $\mathbb{P}^n$ in general position be fixed: $P_0, \dots, P_{n+1}$ and $Q_0, \dots, Q_{n+1}$. Then there exists one, and only one, isomorphic projective transformation τ of $\mathbb{P}^n$ in itself, such that $\tau(P_i) = Q_i$ for all index i.

Proof. Put $P_i = [v_i]$, $Q_i = [w_i]$, $i = 0, \dots, n + 1$. So $\{v_0, \dots, v_n\}$ and $\{w_0, \dots, w_n\}$ are two bases of K^{n+1}, hence there exist scalars $\lambda_0, \dots, \lambda_n, \mu_0, \dots, \mu_n$ such that

$$v_{n+1} = \lambda_0 v_0 + \cdots + \lambda_n v_n, \quad w_{n+1} = \mu_0 w_0 + \cdots + \mu_n w_n,$$

where the coefficients are all different from 0, because of the general position assumption. We replace v_i with $\lambda_i v_i$ and w_i with $\mu_i w_i$ and get two new bases, so there exists a unique automorphism of K^{n+1} transforming the first basis in the second one and, by consequence, also v_{n+1} in w_{n+1}. This automorphism induces the required projective transformation on $\mathbb{P}^n$.
□

An immediate consequence of the above theorem is that projective subspaces of the same dimension are projectively equivalent. Also two subsets of $\mathbb{P}^n$ formed both by k points in general position are projectively equivalent if $k \leq n+2$. If $k > n+2$, this is no longer true, already in the case of four points on a projective line. The problem of describing the classes of projective equivalence of k–tuples of points of $\mathbb{P}^n$, for $k > n+2$, is one the first problems of classical Invariant Theory. The solution in the case $k = 4$, $n = 1$ is given by the notion of *cross–ratio*.

Example 1.10.

Let $X \subset \mathbb{A}^n$ be an affine variety, then $X_F := X \setminus V(F)$ is isomorphic to a closed subset of $\mathbb{A}^{n+1}$, i.e. to $Y = V(x_{n+1}F - 1, G_1, \dots, G_r)$, where $I(X) = \langle G_1, \dots, G_r \rangle$. Indeed, the following regular maps are inverse each other:

- $\varphi : X_F \to Y$ such that $(x_1, \dots, x_n) \to (x_1, \dots, x_n, 1/F(x_1, \dots, x_n))$,
- $\psi : Y \to X_F$ such that $(x_1, \dots, x_n, x_{n+1}) \to (x_1, \dots, x_n)$.

Hence, X_F is a quasi–projective variety contained in $\mathbb{A}^n$, not closed in $\mathbb{A}^n$, but isomorphic to a closed subset of another affine space. The affine varieties of the form X_F are called *special affine open sets*.

A nice example of the previous situation is obtained by taking $\mathbb{A}^1 \setminus V(x) = \mathbb{A}^1 \setminus \{0\}$: the affine line deprived of one point. It is isomorphic to $Y = V(xy - 1) \subset \mathbb{A}^2$, the hyperbola.

From now on, the term *affine variety* will denote a *locally closed subset of a projective space isomorphic to some affine closed set.* Note that, by the previous example, the notion of closed affine set is not preserved under isomophism.

If X is an affine variety and precisely $X \simeq Y$, with $Y \subset \mathbb{A}^n$ closed, then $\mathcal{O}(X) \simeq \mathcal{O}(Y) = K[t_1, \dots, t_n]$ is a finitely generated K–algebra. In particular, since K is algebraically closed, if α is an ideal strictly contained in $\mathcal{O}(X)$, then $V(\alpha) \subset X$ is non–empty, by the relative form of the Nullstellensatz. From this observation, we can deduce that the quasi–projective variety of next example is not affine.

Example 1.11. $\mathbb{A}^2 \setminus \{(0,0)\}$ *is not affine.*

Set $X = \mathbb{A}^2 \setminus \{(0,0)\}$: first of all we will prove that $\mathcal{O}(X) \simeq K[x,y] = \mathcal{O}(\mathbb{A}^2)$, i.e. any regular function on X can be extended to a regular function on the whole plane.

Indeed: let $f \in \mathcal{O}(X)$: if $P \neq Q$ are points of X, then there exist polynomials F, G, F', G' such that $f = F/G$ on a neighbourhood U_P of P and $f = F'/G'$ on a neighbourhood U_Q of Q. So $F'G = FG'$ on $U_P \cap U_Q \neq \emptyset$, which is open also in $\mathbb{A}^2$, hence dense. Therefore $F'G = FG'$ in $K[x,y]$. We can clearly assume that F and G are coprime and similarly for

F' and G'. So by the unique factorization property, it follows that $F' = F$ and $G' = G$. In particular f admits a unique representation as F/G on X and $G(P) \neq 0$ for all $P \in X$. Hence G has no zeros on $\mathbb{A}^2$, so $G = c \in K^*$ and $f \in \mathcal{O}(\mathbb{A}^2)$.

Now, the ideal $\langle x, y \rangle$ has no zeros in X and is proper: this proves that X is not affine.

We have exploited the fact that a polynomial in more than one variables has infinitely many zeros, a fact that allows to generalise the previous observation.

On the other hand, the following property holds:

Proposition 1.12. *Let $X \subset \mathbb{P}^n$ be quasi–projective. Then X admits an open covering by affine varieties.*

Proof. Let $X = X_0 \cup \cdots \cup X_n$ be the open covering of X where $X_i = U_i \cap X = \{P \in X \mid P[a_0, \dots, a_n], a_i \neq 0\}$. So, fixed P, there exists an index i such that $P \in X_i$. We can assume that $P \in X_0$: X_0 is open in some affine variety Y of $\mathbb{A}^n$ (identified with U_0); set $X_0 = Y \setminus Y'$, where Y, Y' are both closed. Since $P \notin Y'$, there exists F such that $F(P) \neq 0$ and $V(F) \supset Y'$. So $P \in Y \setminus V(F) \subset Y \setminus Y'$ and $Y \setminus V(F)$ is an affine open neighbourhood of P in $Y \setminus Y' = X_0 \subset X$. □

As a consequence of Proposition 1.12 we have that, when dealing with local properties, we can reduce ourselves to the case of an affine variety. For instance, the local ring of a point P on a variety X, $\mathcal{O}_{P,X}$, can be replaced by $\mathcal{O}_{P,U}$, where U is an open affine neighborhood of P.

Example 1.13. *The Veronese maps.*

Let n, d be positive integers; put $N(n,d) = \binom{n+d}{d} - 1$. Note that $\binom{n+d}{d}$ is equal to the number of (monic) monomials of degree d in the variables $x_0, \dots, x_n$, that is equal to the number of $(n+1)$–tuples $(i_0, \dots, i_n)$ such that $i_0 + \cdots + i_n = d$, $i_j \geq 0$. Then in $\mathbb{P}^{N(n,d)}$ we can use coordinates $\{v_{i_0 \dots i_n}\}$, where $i_0, \dots, i_n \geq 0$ and $i_0 + \cdots + i_n = d$. For example: if $n = 2$, $d = 2$, then $N(2,2) = \binom{4}{2} - 1 = 5$. In $\mathbb{P}^5$ we can use coordinates $v_{200}, v_{110}, v_{101}, v_{020}, v_{011}, v_{002}$.

For all n, d we define the map $v_{n,d} : \mathbb{P}^n \to \mathbb{P}^{N(n,d)}$ such that

$$[x_0, \dots, x_n] \to [v_{d00\dots0}, v_{d-1,10\dots0}, \dots, v_{0\dots00d}]$$

where $v_{i_0 \dots i_n} = x_0^{i_0} x_1^{i_1} \dots x_n^{i_n}$: $v_{n,d}$ is clearly a morphism, its image is denoted by $V_{n,d}$ and is called *the Veronese variety* of type (n,d). It is in fact the projective variety of equations:

$$(2) \qquad \{v_{i_0 \dots i_n} v_{j_0 \dots j_n} - v_{h_0 \dots h_n} v_{k_0 \dots k_n} = 0, \ \forall i_0 + j_0 = h_0 + k_0, i_1 + j_1 = h_1 + k_1, \dots$$

We prove this statement in the particular case $n = d = 2$; the general case is similar.

First of all, it is clear that the points of $v_{n,d}(\mathbb{P}^n)$ satisfy the system (2). Conversely, assume that $P[v_{200}, v_{110}, \ldots] \in \mathbb{P}^5$ satisfies equations (2), which become:

$$(3)\qquad \begin{cases} v_{200}v_{020} = v_{110}^2 \\ v_{200}v_{002} = v_{101}^2 \\ v_{002}v_{020} = v_{011}^2 \\ v_{200}v_{011} = v_{110}v_{101} \\ v_{020}v_{101} = v_{110}v_{011} \\ v_{110}v_{002} = v_{011}v_{101} \end{cases}$$

Then, at least one of the coordinates $v_{200}, v_{020}, v_{002}$ is different from 0.

Therefore, if $v_{200} \neq 0$, then $P = v_{2,2}([v_{200}, v_{110}, v_{101}])$; if $v_{020} \neq 0$, then $P = v_{2,2}([v_{110}, v_{020}, v_{011}])$; if $v_{002} \neq 0$, then $P = v_{2,2}([v_{101}, v_{011}, v_{002}])$. Note that, if two of these three coordinates are different from 0, then the points of $\mathbb{P}^2$ found in this way have proportional coordinates, so they coincide.

We have also proved in this way that $v_{2,2}$ is an isomorphism between $\mathbb{P}^2$ and $V_{2,2}$, called the Veronese surface of $\mathbb{P}^5$. The same happens in the general case.

Note that equations (3) can be interpreted as the 2×2 minors of the symmetric matrix

$$M = \begin{pmatrix} v_{200} & v_{110} & v_{101} \\ v_{110} & v_{020} & v_{011} \\ v_{101} & v_{011} & v_{002} \end{pmatrix}.$$

So a point $P = [v_{200}.v_{110}, \ldots]$ of $\mathbb{P}^5$ belongs to the Veronese surface $V_{2,2}$ if and only if the rank of this matrix is < 2, i.e. the three lines are proportional. They represent the unique point of $\mathbb{P}^2$ whose image in $v_{2,2}$ is P. So the inverse map $v_{2,2}^{-1} : V_{2,2} \to \mathbb{P}^2$ has three possible expressions by homogeneous polynomials of degree 1, corresponding to the rows of M.

If $n = 1$, $v_{1,d} : \mathbb{P}^1 \to \mathbb{P}^d$ takes $[x_0, x_1]$ to $[x_0^d, x_0^{d-1}x_1, \ldots, x_1^d]$: the image is called the *rational normal curve* of degree d, it is isomorphic to $\mathbb{P}^1$. If $d = 3$, we find the skew cubic.

Let now $X \subset \mathbb{P}^n$ be a hypersurface of degree d: $X = V_P(F)$, with

$$F = \sum_{i_0+\cdots+i_n=d} a_{i_0\ldots i_n} x_0^{i_0} \ldots x_n^{i_n}.$$

Then $v_{n,d}(X) \simeq X$: it is the set of points

$$\{v_{i_0\ldots i_n} \in \mathbb{P}^{N(n,d)} | \sum_{i_0+\cdots+i_n=d} a_{i_0\ldots i_n} v_{i_0\ldots i_n} = 0 \text{ and } [v_{i_0\ldots i_n}] \in V_{n,d}\}.$$

It coincides with $V_{n,d} \cap H$, where H is a hyperplane of $\mathbb{P}^{N(n,d)}$: a hyperplane section of the Veronese variety. This is called the linearisation process, allowing to "transform" a hypersurface in a hyperplane, modulo the Veronese isomorphism.

The Veronese surface $V = V_{2,2}$ of $\mathbb{P}^5$ enjoys a lot of interesting properties. Most of them follow from its property of being covered by a 2-dimensional family of conics, which are precisely the images via $v_{2,2}$ of the lines of the plane.

To see this, we will change notation and will use as coordinates in $\mathbb{P}^5$ $w_{00}, w_{01}, w_{02}, w_{11}, w_{12}, w_{22}$, so that $v_{2,2}$ sends $[x_0, x_1, x_2]$ to the point of coordinates $w_{ij} = x_i x_j$. With this choice of coordinates, the equations of V are obtained by annihilating the 2×2 minors of the symmetric matrix:

$$M' = \begin{pmatrix} w_{00} & w_{01} & w_{02} \\ w_{01} & w_{11} & w_{12} \\ w_{02} & w_{12} & w_{22} \end{pmatrix}.$$

Let ℓ be a line of $\mathbb{P}^2$ of equation $b_0x_0 + b_1x_1 + b_2x_2 = 0$. Its image is the set of points of $\mathbb{P}^5$ with coordinates $w_{ij} = x_i x_j$, such that there exists a non-zero triple $[x_0, x_1, x_2]$ with $b_0x_0 + b_1x_1 + b_2x_2 = 0$. But this last equation is equivalent to the system:

$$\begin{cases} b_0x_0^2 + b_1x_0x_1 + b_2x_0x_2 = 0 \\ b_0x_0x_1 + b_1x_1^2 + b_2x_1x_2 = 0 \\ b_0x_0x_2 + b_1x_1x_2 + b_2x_2^2 = 0 \end{cases}$$

It represents the intersection of V with the plane

$$\begin{cases} b_0w_{00} + b_1w_{01} + b_2w_{02} = 0 \\ b_0w_{01} + b_1w_{11} + b_2w_{12} = 0 \\ b_0w_{02} + b_1w_{12} + b_2w_{22} = 0 \end{cases} \tag{4}$$

so $v_{2,2}(\ell)$ is a plane curve. Its degree is the number of points in its intersection with a general hyperplane in $\mathbb{P}^5$: this corresponds to the intersection in $\mathbb{P}^2$ of ℓ with a conic (a hypersurface of degree 2). Therefore $v_{2,2}(\ell)$ is a conic.

So the isomorphism $v_{2,2}$ transforms the geometry of the lines in the plane in the geometry of the conics in the Veronese surface. In particular, given two distinct points on V, there is exactly one conic contained in V and passing through them.

From this observation it is easy to deduce that the *secant lines* of V, i.e. the lines meeting V at two points, are precisely the lines of the planes generated by the conics contained in V, so that the (closure of the) union of these secant lines coincides with the union of the planes of the conics of V. This union results to be the cubic hypersurface defined by the equation

$$\det M' = \det \begin{pmatrix} w_{00} & w_{01} & w_{02} \\ w_{01} & w_{11} & w_{12} \\ w_{02} & w_{12} & w_{22} \end{pmatrix} = 0.$$

Indeed a point in $\mathbb{P}^5$, of coordinates $[w_{ij}]$ belongs to the plane of a conic contained in V if and only if there exists a non-zero triple $[b_0, b_1, b_2]$ which is solution of the homogeneous system (4).

Exercises 1.14. 1. Let X, Y be closed subsets of $\mathbb{A}^n$. Consider $X \times Y \subset \mathbb{A}^{2n}$ and the linear subspace, called the diagonal, $\Delta \subset \mathbb{A}^{2n}$ defined by the equations $x_i - y_i = 0$, $i = 1, \dots, n$. Prove that $(X \times Y) \cap \Delta$ is isomorphic to $X \cap Y$, constructing an explicit regular map with regular inverse.

2. Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be the map defined by $f(x, y) = (x, xy)$. Check that f is regular and find the image $f(\mathbb{A}^2)$: is it open in $\mathbb{A}^2$? Dense? Closed? Locally closed? Irreducible?

3. Let $v_{1,d} : \mathbb{P}^1 \to \mathbb{P}^d$ be the d-tuple Veronese map, such that $v_{1,d}([x_0, x_1]) = [x_0^d, x_0^{d-1}x_1, \dots, x_1^d])$.

a) Check that the image of $v_{1,d}$ is C_d, the projective algebraic set defined by the 2×2 minors of the matrix

$$A = \begin{pmatrix} z_0 & z_1 & \dots & z_{d-1} \\ z_1 & z_2 & \dots & z_d \end{pmatrix}.$$

C_d is called the rational normal curve of degree d.

b) Prove that $v_{1,d} : \mathbb{P}^1 \to C_d$ is an isomorphism, by explicitly constructing its inverse morphism.

c) Prove that any $d+1$ points on C_d are linearly independent in $\mathbb{P}^d$ (Hint: Vandermonde).

Solution of Exercise 3. This exercise generalises the example of the skew cubic.

a) Let $z_0, \dots, z_d$ be coordinates in $\mathbb{P}^d$, so that the image of the Veronese map $v_{1,d}$ is given in parametric form by $z_0 = x_0^d, \dots, z_i = x_0^{d-i}x_1^i, \dots, z_d = x_1^d$. Let I be the ideal generated by the 2×2 minors of A. It is clear that the two rows of the matrix

$$\begin{pmatrix} x_0^d & x_0^{d-1}x_1 & \dots & x_0x_1^{d-1} \\ x_0^{d-1}x_1 & x_0^{d-2}x_1^2 & \dots & x_1^d \end{pmatrix}$$

are proportional for any x_0, x_1, so $v_{1,d}(\mathbb{P}^1) \subset V_P(I) = C_d$.

Conversely, let $[\bar{z}_0, \dots, \bar{z}_d] \in V_P(I)$. We observe that either $\bar{z}_0 \neq 0$ or $\bar{z}_d \neq 0$. If $\bar{z}_0 \neq 0$, then we can multiply all coordinates by $\bar{z}_0{}^{d-1}$ and we get:

$$[\bar{z}_0, \dots, \bar{z}_d] = [\bar{z}_0{}^d, \bar{z}_0{}^{d-1}\bar{z}_1, \dots, \bar{z}_0{}^{d-1}\bar{z}_i, \dots, \bar{z}_0{}^{d-1}\bar{z}_d].$$

If we can prove that $\bar{z}_0{}^{d-1}\bar{z}_i = \bar{z}_0{}^{d-i}\bar{z}_1{}^i$, then we conclude that our point is equal to $v_{1,d}([\bar{z}_0, \bar{z}_1])$. Note that $\bar{z}_0\bar{z}_k = \bar{z}_1\bar{z}_{k-1}$, for any $k = 1, \dots, d$. So $\bar{z}_0{}^{d-1}\bar{z}_i = \bar{z}_0{}^{d-2}(\bar{z}_1\bar{z}_{i-1}) = \bar{z}_0{}^{d-3}\bar{z}_1(\bar{z}_1\bar{z}_{i-2}) = \dots = \bar{z}_0{}^{d-i}\bar{z}_1{}^i$, as wanted.

If instead $\bar{z}_d \neq 0$, proceeding in a similar way we prove that $[\bar{z}_0, \dots, \bar{z}_d] = v_{1,d}([\bar{z}_{d-1}, \bar{z}_d])$.

b) The inverse map $\varphi : C_d \to \mathbb{P}^1$ operates in this way: $\varphi([z_0, \dots, z_d]) = [z_0, z_1] = [z_1, z_2] = \dots = [z_{d-1}, z_d]$. It is well defined because the columns of A are proportional, and it is regular because it is a projection.

c) Let $[z_0^{(k)}, \dots, z_d^{(k)}] = v_{1,d}([x_0^{(k)}, x_1^{(k)}])$, $k = 0, \dots d$, be $d+1$ points on C_d. Let $M = (z_i^{(j)})_{i,j=0,\dots d}$ be the matrix of their coordinates. If $x_0^{(k)} \neq 0$ for any k, we can assume $x_0^{(k)} = 1$ and

$$M = \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1^{(0)} & x_1^{(1)} & \dots & x_1^{(d)} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ {x_1^{(0)}}^d & {x_1^{(1)}}^d & \dots & {x_1^{(d)}}^d \end{pmatrix}.$$

This is a Vandermonde matrix whose determinant is different from zero because the points are distinct.

If one of the points has the first coordinate equal to zero, then it is $[0, 0, \dots, 0, 1]$, so we can assume that it is the first point, and that all the other d points have $x_0^{(k)} = 1$. Therefore

$$M = \begin{pmatrix} 0 & 1 & \dots & 1 \\ 0 & x_1^{(1)} & \dots & x_1^{(d)} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 1 & {x_1^{(1)}}^d & \dots & {x_1^{(d)}}^d \end{pmatrix}.$$

Developing the determinant according to the first column, we find again a Vandermonde determinant, which is different from 0.