Optimization: definitions

The optimization problems we study take the form

$$\max_x f(x) \text{ subject to } x \in S$$

where:

- f the **objective function**,
- x the **choice variable**, and
- S the **constraint set** or **opportunity set**.

Definition

The value x^* of the variable x solves the problem

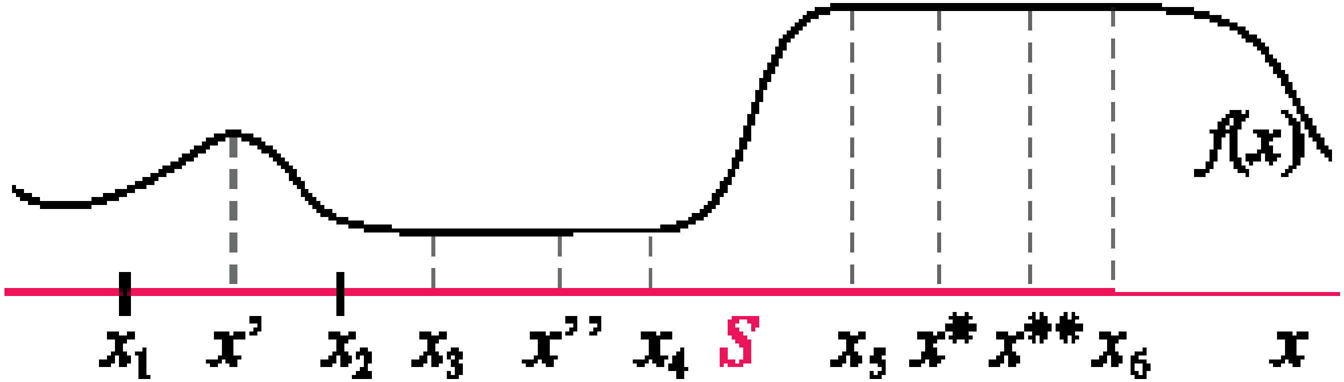
$$\max_x f(x) \text{ subject to } x \in S$$

if $f(x) \leq f(x^*)$ for all $x \in S$.

In this case we say that:

- x^* is a **maximizer** of the function f subject to the constraint $x \in S$
- $f(x^*)$ is the **maximum** (or **maximum value**) of the function f subject to the constraint $x \in S$.

A **minimizer** is defined analogously



x* and ***x***** are **maximizers** of f subject to the constraint $x \in S$
x'' is a **minimizer**

What is ***x'*** ?

It is not a maximizer, because $f(x^*) > f(x')$,
It is not a minimizer, because $f(x'') < f(x')$

But it is a maximum *among the points close to it*. We call such a point a **<u>local maximizer</u>**

UNCONSTRAINED OPTIMIZATION WITH ONE VARIABLE

Necessary conditions

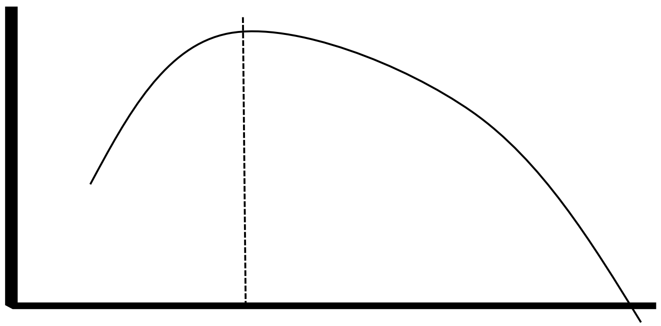
Consider the following problem where $f(x)$ is a differentiable function defined on $\mathbb{R}$

$$\max_x f(x) \text{ subject to } x \in \mathbb{R}$$

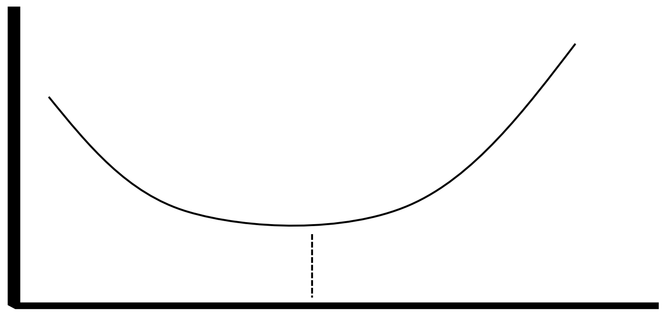
(Remember: $\mathbb{R}$ is the set of real numbers, then x is a single variable)

A point x such that $f'(x) = 0$ is called *stationary point*

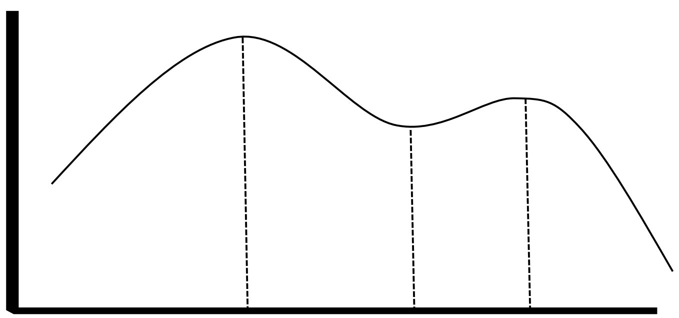
The stationary point is unique and it is maximum



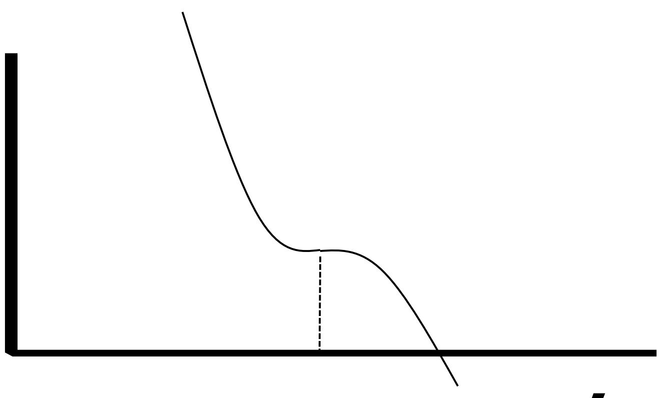
The stationary point is unique and it is minimum



There are three stationary points. No unique solution to the first order conditions



Stationary point is unique and is not a maximum, is not a minimum



From the previous figures we see that:

a stationary point is not necessarily a global or local maximizer, or a global or local minimizer)

a global or local maximizer and a global or local minimizer is necessarily a stationary point

Proposition:

Let f be a differentiable function of a single variable defined on the set of real numbers.

If a point x is a local or global maximizer or minimizer of f then $f'(x) = 0$.

It is a *necessary* condition for x to be a maximizer (or a minimizer) of f:

if x is a maximizer (or a minimizer) **then** x is stationary point of f .

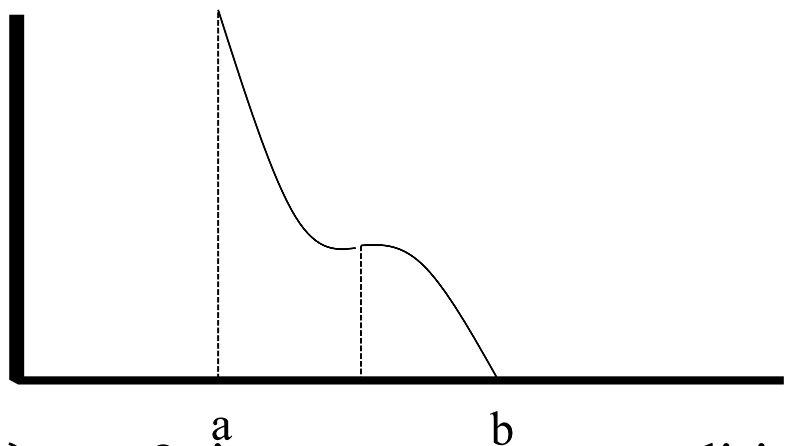
It is **not** *sufficient* for a point to be a maximizer—the condition is satisfied also, for example, at points that are minimizers.

We refer to this condition as a **first-order condition**

Disgression: interior optimum

if we consider the problem: $\max_x f(x)$ subject to $x \in I$ where I is an interval of real numbers, i.e. $I = [a, b]$

In this case a maximum is not necessarily a stationary point



In this case $f'(x) = 0$ is a necessary condition for maximizers and minimizers that are in the interior of I (it means that they are not on the boundaries of I)

If $I = \mathbb{R}$ we are in the previous case, all points are interiors because there is not boundaries.

Example: Consider the problem

$\max_x x^2$ subject to $x \in [-1, 2]$.

This problem satisfies the conditions of the extreme value theorem, and hence has a solution.

Let $f(x) = x^2$. We have $f'(x) = 2x$, so the function has a single stationary point, $x = 0$, which is in the constraint set.

The value of the function at this point is $f(0) = 0$.

The values of f at the endpoints of the interval on which it is defined are $f(-1) = 1$ and $f(2) = 4$.

Thus the global maximizer of the function on $[-1, 2]$ is $x = 2$ and the global minimizer is $x = 0$.

Now we need to study the conditions that allow us to say if a stationary point is an optimizer or not.

Conditions under which a stationary point is a global optimum

Let f be a differentiable function defined on the interval I, and let x be in the interior of I. Then

- if f is concave then x is a global maximizer of f in I if and only if x is a stationary point of f
- if f is convex then x is a global minimizer of f in I if and only if x is a stationary point of f.

Note: a twice-differentiable function is

- concave if and only if its second derivative is non-positive
- convex if and only if its second derivative is non-negatve

Second derivative: it is the derivative of the first derivative

Example: $f(x) = x^3$ the first derivative is $f'(x) = 3x^2$ and the second derivatives is $f''(x) = 6x$

We can denote first and second order derivatives as, respectively, f_x and f_{xx}

Conditions under which a stationary point is a global optimum

Note: a twice-differentiable function is concave if and only if its second derivative is nonpositive (and similarly for a convex function),

Then if f is a twice-differentiable function defined on the interval I and x^* is in the interior of I then:

- $f''(x) \leq 0$ for all $x \in I \Rightarrow$ [x is a global maximizer of f in I if and only if $f'(x^*) = 0$]
- $f''(\mathrm{x}) \geq 0$ for all $x \in I \Rightarrow$ [x is a global minimizer of f in I if and only if $f'(x^*) = 0$].

Example 1: What output maximizes a monopolist's profit?

What quantity maximises consumer utility?

Suppose $f(x) = -x^2 + x - 10$

Three steps:

1. First order condition
2. Solve equation
3. Check that it is a maximum (second order condition < 0)

1. $f_x = -2x + 1 = 0$
2. Solving: $1 = 2x$ so $x = 0.5$
3. $f_{xx} = -2 < 0$ so $x = 0.5$ gives a maximum

Unconstrained Optimization With Many Variables

Consider the problem:

$$max_x\ f(x)\ subject\ to\ x\ \in S$$

where x is a vector

Proposition (First Order Conditions, FOC)

Let f be a differentiable function of n variables defined on the set S. If the point x in the interior of S is a local or global maximizer or minimizer of f then

$$f_i'(x) = 0\ for\ i = 1, \ldots, n.$$

Then the condition that all partial derivatives are equal to zero is a **necessary condition** for an interior optimum (and therefore for an optimum in an unconstrained optimization where each element of x could be any of the real numbers.

Conditions under which a stationary point is a global optimum

Suppose that the function f has continuous partial derivatives in a convex set S and x^* is a stationary point of f in the interior of S (so that $f_i'(x^*) = 0$ for all i).

1. if f is concave then x^* is a global maximizer of f in S **if and only if** it is a stationary point of f

2. if f is convex then x^* is a global minimizer of f in S **if and only if** it is a stationary point of f .

Note to check concavity/convexity we need to compute the second derivatives

Note on derivatives.

Function of two variable $f(x, y)$

There are two first order derivatives:

$f_x(x, y) = \frac{df(x,y)}{dx}$ and $f_y(x, y) = \frac{df(x,y)}{dy}$

There are three second derivatives (derivative of a derivative):

$$f_{xx} = f_{xx}(x, y) = \frac{df_x(x,y)}{dx},$$

$$f_{yy} = f_{yy}(x, y) = \frac{df_y(x,y)}{dy}$$

$$f_{xy} = f_{yx} = f_{xy}(x, y) = f_{yx}(x, y) = \frac{df_x(x,y)}{dy} = \frac{df_y(x,y)}{dx}$$

Unconstrained Maximization with two variables

For example Utility = U(x, y) or Output = F(K, L)

Now try to find the values of x and y which maximize a function $f(x, y)$

Three steps:

1. Set **both** 1st order conditions equal to zero $f_x = 0$ and $f_y = 0$

(the slope of the function with respect to both variables must be simultaneously zero)

2. Solve the equations simultaneously for x and y

However this is a necessary but not sufficient condition (saddle points, minimum points,….)

3. Check second order conditions

Second order conditions

Second order conditions (for maximization)

$$f_{xx} < 0, \; f_{yy} < 0 \text{ and } \quad f_{xx}f_{yy} - f^2_{xy} > 0$$

If this set of condition is satisfied we can say that function f is concave

Note: Second order conditions (for minimization) are

$$f_{xx} > 0, \; f_{yy} > 0 \text{ and } \quad f_{xx}f_{yy} - f^2_{xy} > 0$$

(Function f is convex)

$f(x,y) = 4x - 2x^2 + 2xy - y^2$

1. (i). $f_x = 4 - 4x + 2y = 0$

   (ii). $f_y = 2x - 2y = 0$

2. Solve: from (ii) we have $x = y$

   insert into (i) to get $4 - 4x + 2x = 0$ or

   $4 = 2x$ or $x = 2$

   so $y = x = 2$

3. $f_{xx} = -4 < 0$, $f_{yy} = -2 < 0$

$f_{xy} = f_{yx} = 2$

$$f_{xx}f_{yy} - f^2_{xy} = (-4)(-2) - (2)^2 = 4 > 0$$

Then f is (strictly) concave, so we have a maximum point where $x = 2$ and $y = 2$

Example

Total revenue $R = 12q_1 + 18q_2$

Total Cost $= 2q_1^2 + q_1q_2 + 2q_2^2$

Find the values of q_1 and q_2 that maximise profit

Profit = revenue – cost $= 12q_1 + 18q_2 - (2q_1^2 + q_1q_2 + 2q_2^2)$

$$\pi(q_1, q_2) = 12q_1 + 18q_2 - (2q_1^2 + q_1q_2 + 2q_2^2)$$

The first order conditions are:

$$\begin{cases} \dfrac{d\pi}{dq_1} = 12 - 4q_1 - q_2 = 0 \\ \dfrac{d\pi}{dq_2} = 18 - q_1 - 4q_2 = 0 \end{cases}$$

Solving for q_1 and q_2 gives $q_1 = 2$ and $q_2 = 4$

Is this a maximum? –it will be if function is concave

The second derivatives are:

$\pi_{q_1 q_1} = -4 < 0, \pi_{q_1 q_2} = -4 < 0,$

$\pi_{q_1 q_2} = \pi_{q_2 q_1} = -1$

and

$$\pi_{q_1 q_1} \cdot \pi_{q_2 q_2} - \left(\pi_{q_1 q_2}\right)^2 = 15 > 0$$

then f is concave and the values for q_1 and q_2 maximise profits