Constrained optimization: equality constraints

A firm chooses output x to maximize a profit function

$$f(x) = -x^2 + 10x - 6$$

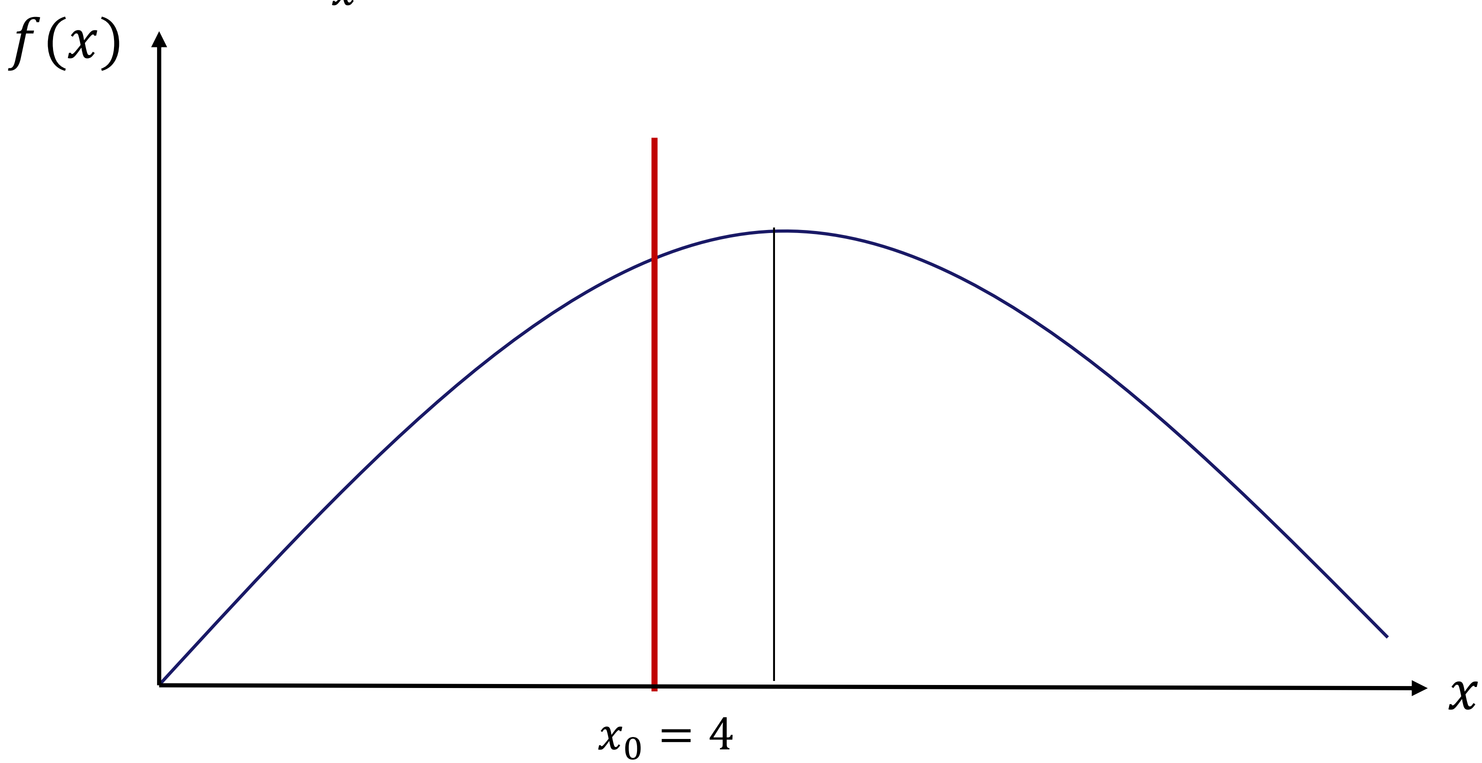
$f(x)$ is the ***objective function.***

Because of a staff shortage, it cannot produce an output higher than $x = x_0$

This gives an ***inequality constraints*** $x \leq x_0$, usually written as $x - x_0 \leq 0$

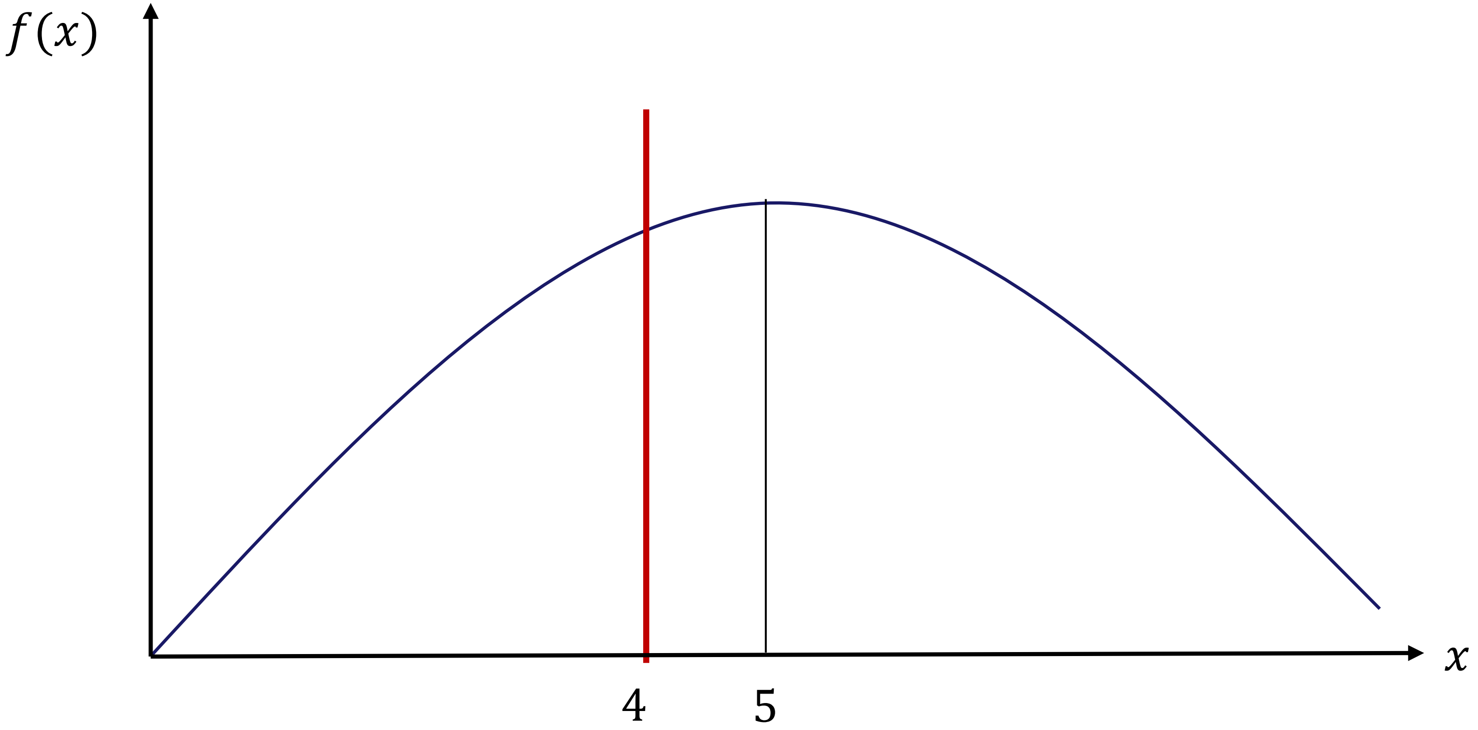
The problem is written as:

$$\max_x -x^2 + 10x - 6 \text{ subject to } x - x_0 \leq 0$$



Note that without the constraint the optimum is $x = 5$

So the constraint is binding (but a constraint of, say, $x \leq 6$ would not be)



Constrained optimization with two variables and one constraint

The problem is:

$$\max_{\{x,y\}} f(x,y)$$

$$s.t\ g(x,y) = c \qquad x, y \in S$$

To get the solution we have to write the *Lagrangean:*

$$L(x,y,\lambda) = f(x,y) - \lambda(g(x,y) - c)$$

where λ is a new variable

The candidates to the solution are the stationary points of the Lagrangean, i.e. all points that satisfy the following system of equations:

$$\begin{cases} f'_x(x,y) - \lambda g'_x(x,y) = 0 \\ f'_y(x,y) - \lambda g'_y(x,y) = 0 \\ g(x,y) = c \end{cases}$$

Necessary conditions for an optimum

Let f and g be continuously differentiable functions of two variables defined on the set S, c be a number.

Suppose that:

- (x^*, y^*) is an interior point of S that solves the problem

$$\max_{\{x,y\}} f(x,y) \quad s.t\ g(x,y) = c \qquad x, y \in S$$

- $either\, g_1'(x^*, y^*) \neq 0 \; or\, g_2'(x^*, y^*) \neq 0.$

Then there is a unique number λ such that (x^*, y^*) is a **stationary point** of the Lagrangean

$$L(x, y) = f(x, y) - \lambda(g(x, y) - c)$$

That is, (x^*, y^*) satisfies the first-order conditions.

$$L_1(x^*, y^*) = f_1'(x^*, y^*) - \lambda g_1'(x^*, y^*) = 0$$
$$L_2(x^*, y^*) = f_2'(x^*, y^*) - \lambda g_2'(x^*, y^*) = 0.$$
$$\text{and } g(x^*, y^*) = c.$$

Interpretation of λ

Suppose we solve the problem

$$\max_{\{x,y\}} f(x,y) \quad s.t\ g(x,y)=c$$

then the solution are function of parameter c, i.e. $\lambda^*(c)$, $x^*(c)$, $y^*(c)$.

Also suppose that FOCs hold.

The maximum value of the objective function is also a function of c:

$$f^*(c) = f^*(x^*(c)\ , y^*(c))$$

Then

$$\frac{\partial f^*(c)}{\partial c} = \ \lambda^*(c)$$

the value of the Lagrange multiplier at the solution of the problem is equal to the rate of change in the maximal value of the objective function as the constraint is relaxed.

It is the shadow price of the constrained resource

Example 2:

$$\max_{\{x\}} x^2$$
$$s.t. \quad x = c$$

solution is $x = c$ so the maximized value of the objective function is c^2.

Its derivative respect to c is $2c$

Now consider the Lagrangean

$$L(x) = x^2 - \lambda(x - c)$$

The FOC is $2x - \lambda = 0$.

Then $x = c$ and $\lambda = 2c$ satisfy FOC and the constraint.
Note that λ is equal to the derivative of the maximized value of the function with respect to c

From example 1:

The solution is:

$$x = \frac{c\ a}{a+b} \qquad y = \frac{c\ b}{a+b} \qquad \lambda = \frac{a^a b^b}{(a+b)^{a+b-1}} \cdot c^{a+b-1}$$

the maximized value of the objective function is:

$$x^a y^b = \frac{a^a b^b}{(a+b)^{a+b}} \cdot c^{a+b}$$

its derivative respect to c is:

$$\frac{a^a b^b}{(a+b)^{a+b-1}} \cdot c^{a+b-1}, \text{ i.e. } \lambda$$

Conditions under which a stationary point is a global optimum

Suppose that f and g are continuously differentiable functions defined on an open convex subset S of two-dimensional space and

suppose that there exists a number λ^* such that (x^*, y^*) is an interior point of S that is a stationary point of the Lagrangean

$$L(x, y) = f(x, y) - \lambda^*(g(x, y) - c).$$

Suppose further that $g(x^*, y^*) = c$.

Then if L is concave then (x^*, y^*) solves the problem

$$\max_{\{x,y\}} f(x, y) \quad s.t\ g(x, y) = c$$

Note:

- L is concave if $L'_{xx} < 0, L'_{yy} < 0\ and\ L'_{xx} \cdot L'_{yy} - (L'_{xy})^2 > 0$
- If the constraint is linear you need to check only the concavity of f

Example

Consider example 3

$$\max_{\{x,y\}} x^3 y \ \ subject\ to\ x + y = 6 \quad x, y > 0$$

We found that the solution of the FOC $x = 4.5, y = 1.5, \lambda = \frac{2}{3}$ is a local maximizer**.**

Is it a global maximizer?

For a global maximizer we need that Lagrangean is concave

$$L(x, y) = 3\ln x + \ln y - \lambda(x + y - 6)$$

Given that constraint is linear we need to check the objective function

Compute the second order derivative of f

$$f'_{xx} = -\frac{3}{x^2}, f'_{yy} = -\frac{1}{y^2}\ and\ \ f'_{xy} = 0$$

Then conditions for concavity $f'_{xx} < 0, f'_{yy} < 0\ and\ f'_{xx} \cdot f'_{yy} - \left(f'_{xy}\right)^2 > 0$ are satisfied

Envelope theorem: unconstrained problem

Let $f(x,r)$ be a continuously differentiable function where x is an *n*-vector of variables and r is a *k*-vector of parameters.

The maximal value of the function is given by $f(x^*(r),r)$ where $x^*(r)$, is the vector of variables x that maximize f and that are function of r.

Note that we can write $f(x^*(r),r)$ as $f^*(r)$ (because in this function only parameters appear)

If the solution of the maximization problem is a continuously differentiable function of *r* then:

$$\frac{df^*(r)}{dr_i} = \frac{df(x,r)}{dr_i} \quad \text{evaluated in } x^*(r),$$

the change in the maximal value of the function as a parameter changes is the change caused by the direct impact of the parameter on the function, holding the value of *x* fixed at its optimal value;

the indirect effect, resulting from the change in the optimal value of *x* caused by a change in the parameter, is zero

Example 6

$$\max\ p \ln x - cx$$

FOC is $\frac{p}{x} - c = 0$

then $x^* = \frac{p}{c}$

and $f^*(p,c) = p \ln \frac{p}{c} - p$

The effect of a change of parameter c on the maximized value is:

$$\frac{df^*(p,c)}{dc} = -\frac{p}{c}$$

Consider the derivative of the objective function evaluated at the solution x^*

$$\frac{dp \ln x - cx}{dc} = -x$$

Evaluating it in $x^* = \frac{p}{c}$ we get $-\frac{p}{c}$

Envelope theorem: constrained problems

Let $f(x, r)$ be a continuously differentiable function where x is an *n*-vector of variables and r is a *k*-vector of parameters.

The maximal value of the function is given by $f(x^*(r), r)$

where $x^*(r)$, is the vector of variables x that maximize f and that are function of r.

Note that we can write $f(x^*(r))$, as $f^*(r)$,

Then

$$\frac{df^*(r)}{dr_i} = \frac{dL\,(\mathrm{x}, r)}{dr_i}$$ evaluated at the solution $x^*(r)$,

where the function L is the Lagrangean of the problem

Example 7 $$\max_{\{x,y\}} xy \;\; s.t\; x + y = B$$

$$L(x, y, \lambda) = xy - \lambda(x + y - B)$$

we solve:

$$y - \lambda = 0$$

$$x - \lambda = 0$$

$$x + y = B$$

then $x^* = y^* = \lambda^* = \frac{B}{2}$ and $f^*(B) = \frac{B^2}{4}$

The effect of a change of parameter c on the maximized value is:

$$\frac{df^*(B)}{dB} = \frac{B}{2}$$

Consider the derivative of the Lagrangean evaluated at the solution x^*

$$\frac{d\big(xy - \lambda(x + y - B)\big)}{dB} = \frac{B}{2}$$