Capital and Time

Capital

The capital stock of an economy is the total sum of machines, buildings and other reproducible resources in existence at a point in time

- these assets represent some part of the economy's past output that was not consumed, but was instead set aside for future production

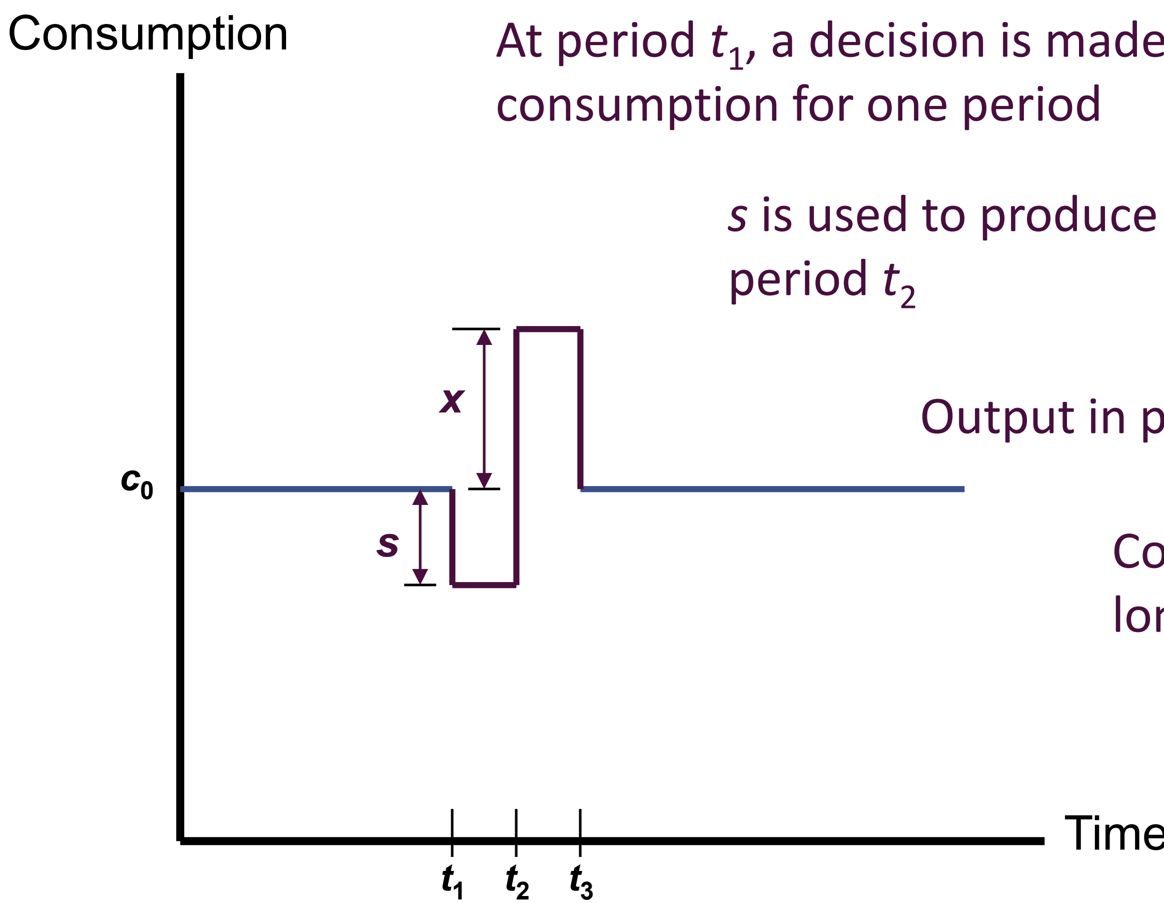
Rate of Return

At period t_1, a decision is made to hold s from current consumption for one period

s is used to produce additional output only in period t_2

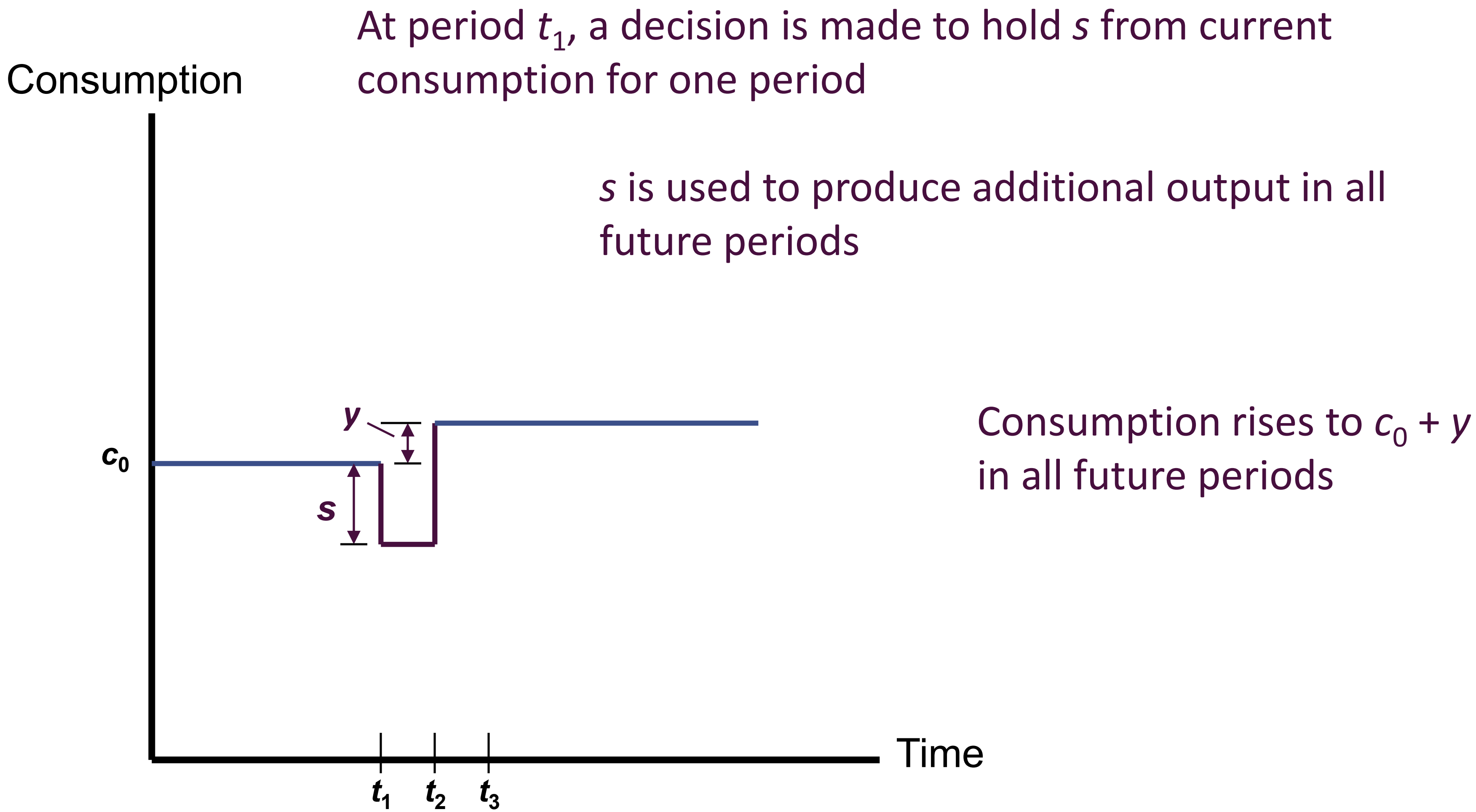
Output in period t_2 rises by x

Consumption returns to its long-run level (c_0) in period t_3



The single period rate of return (r_1) on an investment is the extra consumption provided in period 2 as a fraction of the consumption forgone in period 1

$$r_1 = \frac{x - s}{s} = \frac{x}{s} - 1$$



The perpetual rate of return (r_∞) is the permanent increment to future consumption expressed as a fraction of the initial consumption foregone

$$r_\infty = \frac{y}{s}$$

When economists speak of the rate of return to capital accumulation, they have in mind something between these two extremes

i.e. a measure of the terms at which consumption today may be turned into consumption tomorrow

The main question is:

How the economy’s rate of return is determined?

… the equilibrium arises from the supply and demand for present and future goods

Determine the rate of return

Rate of Return and Price of Future Goods

Assume that there are only two periods: the current period, denoted by 0, and the next period, denoted by 1.

Let Δ to denote the change in consumption: Δc_0 is the consumption forgone in period 0 and Δc_1 is its increase in period 1

Then the rate of return between these two periods (*r*) is defined to be

$$r = \frac{\Delta c_1}{\Delta c_0} - 1$$

Rewriting, we get

$$\frac{\Delta c_0}{\Delta c_1} = \frac{1}{1 + r}$$

The relative price of future goods (p_1) is the quantity of present goods that must be foregone to increase future consumption by one unit

$$p_1 = \frac{\Delta c_0}{\Delta c_1} = \frac{1}{1+r}$$

Demand for Future Goods

An individual's utility depends on present and future consumption

$$U = U(c_0, c_1)$$

and the individual must decide how much current wealth (w) to devote to these two goods

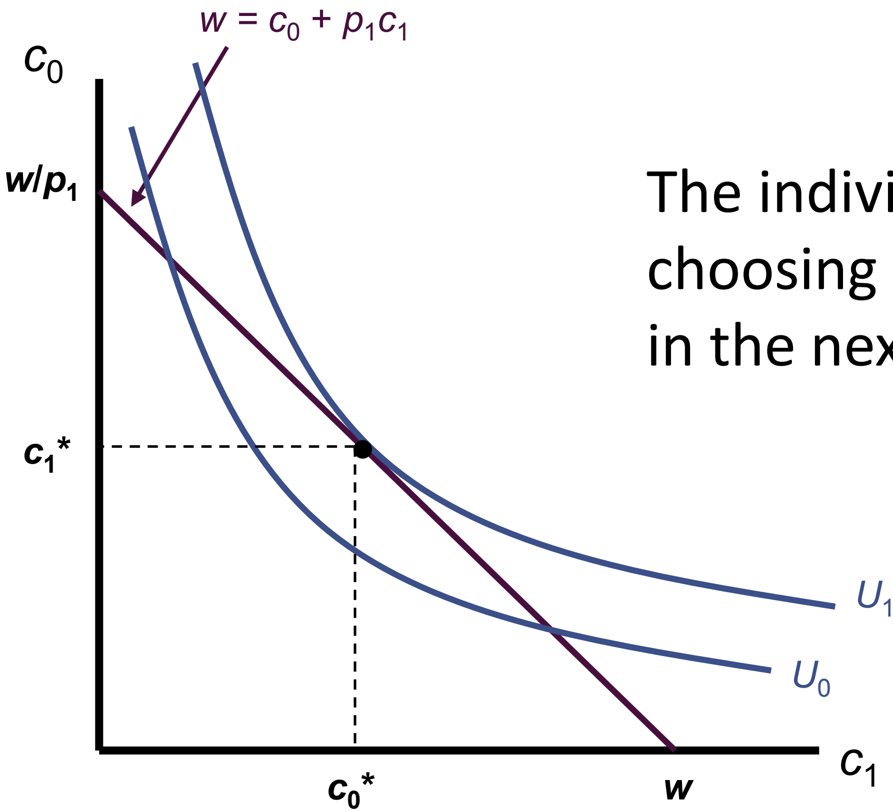
The budget constraint is

$$w = c_0 + p_1 c_1$$

Then the individual's problem is

$$\max_{c_0, c_1} U(c_0, c_1)$$

$$\text{s.t. } w = c_0 + p_1 c_1$$



The individual will maximize utility by choosing to consume c_0^* currently and c_1^* in the next period

The individual consumes c_0^* in the present period and chooses to save $w - c_0^*$ to consume next period

This future consumption can be found from the budget constraint

$$p_1\, c_1^* = w - c_0^*$$
$$c_1^* = (w - c_0^*)/p_1$$
$$c_1^* = (w - c_0^*)(1 + r)$$

Example: Intertemporal Impatience

Individuals' utility-maximizing choices over time will depend on how they feel about waiting for future consumption

Assume that an individual's instantaneous utility function for consumption, $u(c)$, is the same for both periods (with $u' > 0$ and $u'' < 0$) but period 1's utility is discounted by a "rate of time preference" of $\frac{1}{1+\rho}$ (where $\rho > 0$)

This means that

$$U(c_0, c_1) = u(c_0) + \frac{1}{1+\rho} u(c_1)$$

Maximization of this function subject to the intertemporal budget constraint yields the Lagrangian expression

$$L = u(c_0) + \frac{1}{1+\rho} u(c_1) + \lambda \left[w - c_0 - \frac{c_1}{1+r} \right]$$

The first-order conditions for a maximum are:

$$\frac{\partial L}{\partial c_0} = u'(c_0) - \lambda = 0$$

$$\frac{\partial L}{\partial c_1} = \frac{1}{1+\rho} u'(c_1) - \frac{\lambda}{1+r} = 0$$

$$w - c_0 - \frac{c_1}{1+r} = 0$$

Dividing the first and second conditions and rearranging, we find

$$u'(c_0) = \frac{1+r}{1+\rho} u'(c_1)$$

Therefore,

- if $r = \rho$, $c_0 = c_1$
- if $r < \rho$, $c_0 > c_1$
- if $r > \rho$, $c_0 < c_1$

Effects of Changes in *r*

If r rises (and p_1 falls), both income and substitution effects will cause more c_1 to be demanded

- unless c_1 is inferior (unlikely)

This implies that the demand curve for c_1 will be downward sloping

The sign of $\partial c_0/\partial p_1$ is ambiguous

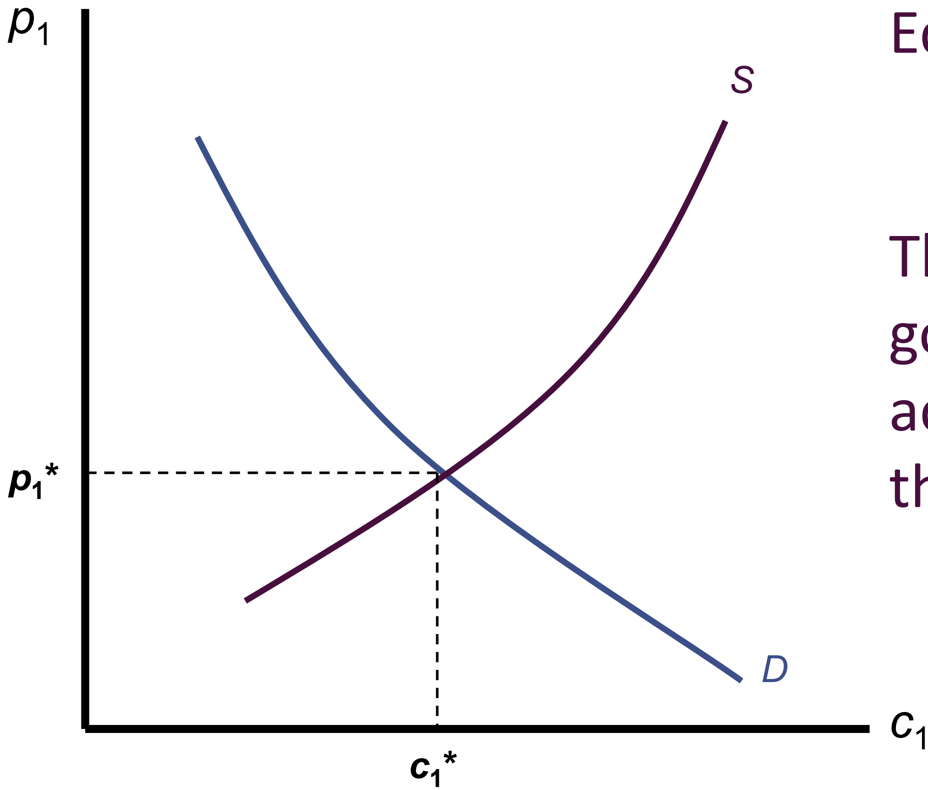
- the substitution and income effects work in opposite directions

Thus, we cannot make an accurate prediction about how a change in the rate of return affects current consumption

Supply of Future Goods

An increase in the relative price of future goods (p_1) will likely induce firms to produce more of them because the yield from doing so is now greater

- this means that the supply curve will be upward sloping



Equilibrium occurs at p_1^* and c_1^*

The required amount of current goods will be put into capital accumulation to produce c_1^* in the future

Equilibrium Price of Future Goods

We expect that $p_1 < 1$

- individuals require some reward for waiting
- capital accumulation is "productive"
  - sacrificing one good today will yield more than one good in the future

The Equilibrium Rate of Return

The price of future goods is

$$p_1^* = \frac{1}{1+r^*} \quad \text{or} \quad r^* = \frac{1-p_1^*}{p_1^*}$$

Because p_1^* is assumed to be < 1, the rate of return (r) will be positive

p_1^* and r are equivalent ways of measuring the terms on which present goods can be turned into future goods

Rate of Return & Real and Nominal Interest Rates

Both the rate of return and the real interest rate refer to the real return that is available from capital accumulation

The nominal interest rate (i) is given by

$$1 + i = (1 + r)(1 + expected\ inflation\ rate)$$

$$1 + i = (1 + r)(1 + \dot{p}_e)$$

Expansion of this equation yields

$$1 + i = 1 + r + \dot{p}_e + r\,\dot{p}_e$$

Assuming that $r\,\dot{p}_e$ is small, i can be approximated by:

$$i = r + \dot{p}_e$$

The Firm's Demand for Capital

In a perfectly competitive market, a firm will choose to hire that number of machines for which the *MRP* is equal to the market rental rate

- Determinant of market rental rates

- Nondepreciating machines

- Ownership of Machines

Determinants of Market Rental Rates

Consider a firm that rents machines to other firms

The owner faces two types of costs:

- depreciation on the machine (assumed to be a constant % d of the machine's market price p)
- the opportunity cost of the funds tied up in the machine rather than another investment (assumed to be the real interest rate r)

The total costs to the machine owner for one period are given by

$$pd + pr = p(r + d)$$

If we assume the machine rental market is perfectly competitive, no long-run profits can be earned renting machines

- the rental rate per period (v) will be equal to the costs

$$v = p(r + d)$$

Nondepreciating Machines

If a machine does not depreciate, $d = 0$ and

$$v/p = r$$

An infinitely long-lived machine is equivalent to a perpetual bond and must yield the market rate of return

Ownership of Machines

Firms commonly own the machines they use

A firm uses capital services to produce output

these services are a flow magnitude (number of machine hours)

It is often assumed that the flow of capital services is proportional to the stock of machines

Demand for capital. A profit-maximizing firm facing a perfectly competitive rental market for capital will hire additional capital up to the point at which the MRP_k is equal to v

- under perfect competition, v will reflect both depreciation costs and the opportunity costs of alternative investments

$$MRP_k = v = p(r + d)$$

Theory of Investment

If a firm decides it needs more capital services that it currently has, it has two options:

- hire more machines in the rental market
- purchase new machinery (it is called investment)

Present Discounted Value

When a firm buys a machine, it is buying a stream of net revenues in future periods

- it must compute the present discounted value of this stream

Consider a firm that is considering the purchase of a machine that is expected to last *n* years

- it will provide the owner monetary returns in each of the n years

The present discounted value (*PDV*) of the net revenue flow from the machine to the owner is given by

$$PDV = \frac{R_1}{1+r} + \frac{R_2}{(1+r)^2} + \cdots + + \frac{R_n}{(1+r)^n}$$

If the *PDV* exceeds the price of the machine, the firm should purchase the machine

Note that this formula is for a firm that computes the PDV of the net revenue flow from the point of view of year 0

In a competitive market, the only equilibrium that can prevail is that in which the price is equal to the *PDV* of the net revenues from the machine

Thus, market equilibrium requires that

$$p = PDV = \frac{R_1}{1+r} + \frac{R_2}{(1+r)^2} + \cdots + + \frac{R_n}{(1+r)^n}$$

Simple case

Suppose that machines are infinitely long-lived and the *MRP* (R_i) is the same in every year

$R_i = v \; \forall i$ in a competitive market

Therefore, the *PDV* from machine ownership is

$$PDV = \frac{v}{1+r} + \frac{v}{(1+r)^2} + \cdots + \frac{v}{(1+r)^n} + \cdots \ldots = v \sum_{i=1}^{\infty} \frac{1}{(1+r)^1} =$$

$$= \frac{v}{r}$$

In equilibrium $p = PDV$ so

$$p = \frac{v}{r} \text{ or } r = \frac{v}{p}$$

General Case

We can generate similar results for the more general case in which the rental rate on machines is not constant over time and in which there is some depreciation

Suppose that the rental rate for a **new** machine at any time s is given by $v(s)$

The machine depreciates at a rate of d

The net rental rate of the machine will decline over time

In year s the net rental rate of an old machine bought in a previous year (t) would be

$$v(s)e^{-d(s-t)}$$

If the firm is considering the purchase of the machine when it is new in year t, it should discount all of these net rental amounts back to that date

The present value of the net rental in year s discounted back to year t is

$$e^{-r(s-t)}v(s)e^{-d(s-t)} = e^{(r+d)t}v(s)e^{-(r+d)s}$$

The present discounted value of a machine bought in year t is therefore the sum (integral) of these present values

$$PDV(t) = \int_t^{\infty} e^{(r+d)t}v(s)e^{-(r+d)s}ds$$

In equilibrium, the price of the machine at time t, $p(t)$, will be equal to this present value

$$p(t) = PDV(t) = \int_t^{\infty} e^{(r+d)t}v(s)e^{-(r+d)s}ds$$

Rewriting, we get

$$p(t) = e^{(r+d)t} \int_t^{\infty} v(s) e^{-(r+d)s} ds$$

Differentiating with respect to *t* yields:

$$\frac{dp(t)}{dt} = (r+d) e^{(r+d)t} \int_t^{\infty} v(s) e^{-(r+d)s} ds - e^{(r+d)t} v(t) e^{-(r+d)t} =$$

$$= (r+d)p(t) - v(t)$$

Hence

$$v(t) = (r+d)p(t) - \frac{dp(t)}{dt}$$

Note, $\frac{dp(t)}{dt}$ represents the capital gains that accrue to the owner of the machine

Example: Cutting Down a Tree

Consider the case of a forester who must decide when to cut down a tree

Suppose that the value of the tree at any time t is given by $f(t)$ [where $f'(t) > 0$ and $f''(t) < 0$] and that l dollars were invested initially as payments to workers who planted the tree

When the tree is planted, the present discounted value of the owner's profits is

$$PDV(t) = e^{-rt} f(t) - l$$

The forester's decision consists of choosing the harvest date, t, to maximize this value, i.e.

$$\max_t e^{-rt} f(t) - l$$

FOC is: $\frac{dPDV(t)}{dt} = -re^{-rt} f(t) + e^{-rt} f'(t) = 0$

Dividing both sides by e^{-rt},

$$f'(t) - r\,f(t) = 0$$

Therefore, $r = \frac{f'(t)}{f(t)}$

Note that l drops out (sunk cost)

The tree should be harvested when r is equal to the proportional growth rate of the tree

Suppose that trees grow according to the equation

$$f(t) = e^{0.4\sqrt{t}}$$

$$r = \frac{f'(t)}{f(t)} = \frac{0.2}{\sqrt{t}}$$

If r = 0.04, then t^* = 25

If r rises to 0.05, then t^* falls to 16