The Discounted Utility (DU) Model

Paul Samuelson (1937) "A Note on Measurement of Utility.“

- generalized model of intertemporal Choice applicable to multiple time periods
- the discount rate r
- intertemporal preferences over consumption profiles $(c_t, \ldots, cT)$
- preferences can be represented by an intertemporal utility function $U_t(ct, \ldots, cT)$

The utility at time t of a flow of goods c_t accruing at times $t, t+1, t+2, \dots\dots T$ is given by:

$$U_t(c_t, \dots \dots c_T) = \sum_{k=0}^{T-t} \delta^k u(c_{t+k})$$

where $\delta = \frac{1}{1+r}$

and

- $u(c_{t+k})$ instantaneous utility function
- δ discount function (or discount factor)
- r discount rate

- Note that $0 < \delta < 1$

Useful results

$$\delta = \frac{1}{1+r} \text{ where } r > 0$$

$$\sum_{t=0}^{\infty} \delta^t = \frac{1}{1-\delta}$$

$$\sum_{t=1}^{\infty} \delta^t = \frac{\delta}{1-\delta}$$

$$\sum_{t=2}^{\infty} \delta^t = \frac{\delta^2}{1-\delta}$$

$$\sum_{t=x}^{\infty} \delta^t = \frac{\delta^x}{1-\delta}$$

$$\sum_{t=0}^{y} \delta^t = \frac{1-\delta^{y+1}}{1-\delta}$$

$$\sum_{t=1}^{y} \delta^t = \frac{\delta-\delta^{y+1}}{1-\delta}$$

$$\sum_{t=2}^{y} \delta^t = \frac{\delta^2-\delta^{y+1}}{1-\delta}$$

$$\sum_{t=x}^{y} \delta^t = \frac{\delta^x-\delta^{y+1}}{1-\delta}$$

Suppose that $\mathrm{r} = 0.05$ then $\delta = \frac{1}{1+r} = 0.95$

Compute the discounted utility of a infinite stream of payments of 64 euro starting at time $t = 1$ assuming $u(x) = \sqrt{x}$

It is important to define in which time we compute the discounted utility

- If we discount at $t = 0$

$$U_0(c_1, \dots \dots c_\infty) = \sum_{k=1}^{\infty} \delta^k \sqrt{64} = 8 \sum_{k=1}^{\infty} 0.95^k = 8 \frac{0.95}{1-0.95} = 152$$

- If we discount at $t = 1$

$$U_1(c_1, \dots \dots c_\infty) = \sum_{k=0}^{\infty} \delta^k \sqrt{64} = 8 \sum_{k=0}^{\infty} 0.95^k = 8 \frac{1}{1-0.95} = 160$$

- If we discount at $t = 0$ and the stream is not an infinite one but end at $t = 10$

$$U_0(c_1, \dots \dots c_{10}) = \sum_{k=1}^{10} \delta^k \sqrt{64} = 8 \sum_{k=1}^{10} 0.95^k = 8 \frac{0.95-0.95^{11}}{1-0.95} = \cdots.$$

when time is continuous, discounted utility use continuous compounding

$$U_t(c_{t+x}) = u(c_{t+x})e^{-x\,r}$$

i.e. the utility at time t of consumption at time $t + x$

The discounted value of a stream of consumption from time t to time $t + x$ is given by

$$U_t(c) = \int_0^x u(c_{t+x})e^{-x\,r}dx$$

From the previous example, if we discount at $t = 1$ the infinite stream

$$U_1(c) = \int_0^\infty \sqrt{64}e^{-x\,0.05}dx = 8\int_0^\infty e^{-x\,0.05}dx = -\frac{1}{0.05}e^{-x\,0.05}\Big|_1^\infty \cdot 8 =$$

$$= 8 \cdot \frac{1}{0.05}e^{-0.05} = 152$$

Suppose that the current value (t=1) of an object you have is 100. This value evolves over the time according the following formula: $v(t) = 100\sqrt{t}$

Compute the optimal time to sell this object assuming that $r = 0.05$, $p(t) = v(t)$ and your utility function is $u(x) = x$

The discounted utility to sell this object at time $t = \tau$ is

$$U_1(\tau) = 100\sqrt{\tau}e^{-0.05(\tau-1)}$$

The problem is

$$\max_{\tau} 100\sqrt{\tau}e^{-0.05(\tau-1)}$$

FOC is $\frac{100}{2\sqrt{\tau}}e^{-0.05(\tau-1)} - 5\sqrt{\tau}e^{-0.05(\tau-1)} = 0 \rightarrow \tau = 10$