# LESSON 17.

## 1. The tangent space and the notion of smoothness.

We will always assume $K$ algebraically closed. In this Lesson we follow the approach of Šafarevič. We define the tangent space $T_{X,P}$ at a point $P$ of an *affine* variety $X \subset \mathbb{A}^n$ as the union of the lines passing through $P$ and "touching" $X$ at $P$. It results to be an affine subspace of $\mathbb{A}^n$. Then we will find a "local" characterization of $T_{X,P}$, this time interpreted as a vector space, the direction of $T_{X,P}$, only depending on the local ring $\mathcal{O}_{X,P}$: this will allow to define the tangent space at a point of any quasi–projective variety.

Assume first that $X \subset \mathbb{A}^n$ is closed and $P = O = (0, \dots, 0)$. Let $L$ be a line through $P$: if $A(a_1, \dots, a_n)$ is another point of $L$, then a general point of $L$ has coordinates $(ta_1, \dots, ta_n)$, $t \in K$. If $I(X) = (F_1, \dots, F_m)$, then the intersection $X \cap L$ is determined by the following system of equations in the indeterminate $t$:

$$F_1(ta_1, \dots, ta_n) = \dots = F_m(ta_1, \dots, ta_n) = 0.$$

The solutions of this system of equations are the roots of the greatest common divisor $G(t)$ of the polynomials $F_1(ta_1, \dots, ta_n), \dots, F_m(ta_1, \dots, ta_n)$ in $K[t]$, i.e. the generator of the ideal they generate. We may factorize $G(t)$ as $G(t) = ct^e(t - \alpha_1)^{e_1} \dots (t - \alpha_s)^{e_s}$, where $c \in K$, $\alpha_1, \dots, \alpha_s \neq 0$, $e, e_1, \dots, e_s$ are non-negative, and $e > 0$ if and only if $P \in X \cap L$. The number $e$ is by definition the **intersection multiplicity at $P$ of $X$ and $L$.** If $G(t)$ is identically zero, then $L \subset X$ and the intersection multiplicity is, by definition, $+\infty$.

Note that the polynomial $G(t)$ doesn't depend on the choice of the generators $F_1, \dots, F_m$ of $I(X)$, but only on the ideal $I(X)$ and on $L$.

**Definition 1.1.** The line $L$ is **tangent to the variety $X$ at $P$** if the intersection multiplicity of $L$ and $X$ at $P$ is at least 2 (in particular, if $L \subset X$). The **tangent space to $X$ at $P$** is the union of the lines that are tangent to $X$ at $P$; it is denoted $T_{P,X}$.

We will see now that $T_{P,X}$ is an affine subspace of $\mathbb{A}^n$. Assume that $P \in X$: then the polynomials $F_i$ may be written in the form $F_i = L_i + G_i$, where $L_i$ is a homogeneous linear polynomial (possibly zero) and $G_i$ contains only terms of degree $\geq 2$. Then

$$F_i(ta_1, \dots, ta_n) = tL_i(a_1, \dots, a_n) + G_i(ta_1, \dots, ta_n),$$

where the last term is divisible by $t^2$. Let $L$ be the line $\overline{OA}$, with $A = (a_1, \dots, a_n)$. We note that the intersection multiplicity of $X$ and $L$ at $P$ is the maximal power of $t$ dividing the greatest common divisor, so $L$ is tangent to $X$ at $P$ if and only if $L_i(a_1, \dots, a_n) = 0$ for all $i = 1, \dots, m$.

Therefore the point $A$ belongs to $T_{P,X}$ if and only if

$$L_1(a_1, \dots, a_n) = \dots = L_m(a_1, \dots, a_n) = 0.$$

This shows that $T_{P,X}$ is a linear subspace of $\mathbb{A}^n$, whose equations are the linear components of the equations defining $X$.

**Example 1.2.** *(i)* $T_{O,\mathbb{A}^n} = \mathbb{A}^n$, *because* $I(\mathbb{A}^n) = (0)$.

*(ii) If* $X$ *is a hypersurface, with* $I(X) = (F)$, *we write as above* $F = L + G$; *then* $T_{O,X} = V(L)$: *so* $T_{O,X}$ *is either a hyperplane if* $L \neq 0$, *or the whole space* $\mathbb{A}^n$ *if* $L = 0$. *For instance, if* $X$ *is the affine plane cuspidal cubic* $V(x^3 - y^2) \subset \mathbb{A}^2$, $T_{O,X} = \mathbb{A}^2$.

Assume now that $P \in X$ has coordinates $(y_1, \dots, y_n)$. With a linear transformation we may translate $P$ to the origin $(0, \dots, 0)$, taking as new coordinates functions on $\mathbb{A}^n$ $x_1 - y_1, \dots, x_n - y_n$. This corresponds to considering the $K$-isomorphism $K[x_1, \dots, x_n] \longrightarrow K[x_1 - y_1, \dots, x_n - y_n]$, which takes a polynomial $F(x_1, \dots, x_n)$ to its Taylor expansion

$$G(x_1 - y_1, \dots, x_n - y_n) = F(y_1, \dots, y_n) + d_P F + d_P^{(2)} F + \dots,$$

where $d_P^{(i)} F$ denotes the $i^{th}$ differential of $F$ at $P$: it is a homogeneous polynomial of degree $i$ in the variables $x_1 - y_1, \dots, x_n - y_n$. In particular the linear term is

$$d_P F = \frac{\partial F}{\partial x_1}(P)(x_1 - y_1) + \dots + \frac{\partial F}{\partial x_n}(P)(x_n - y_n).$$

We get that, if $I(X) = (F_1, \dots, F_m)$, then $T_{P,X}$ is the affine subspace of $\mathbb{A}^n$ defined by the equations

$$d_P F_1 = \dots = d_P F_m = 0.$$

The affine space $\mathbb{A}^n$, which may identified with $K^n$, can be given a natural structure of $K$-vector space with origin $P$, so in a natural way $T_{P,X}$ is a vector subspace (with origin $P$). The functions $x_1 - y_1, \dots, x_n - y_n$ form a basis of the dual space $(K^n)^*$ and their restrictions generate $T^*_{P,X}$. Note moreover that $\dim T^*_{P,X} = k$ if and only if $n - k$ is the maximal number of polynomials linearly independent among $d_P F_1, \dots, d_P F_m$. If $d_P F_1, \dots, d_P F_{n-k}$ are these polynomials, then they form a base of the orthogonal $T^{\perp}_{P,X}$ of the vector space $T_{P,X}$ in $(K^n)^*$, because they vanish on $T_{P,X}$.

Let us define now the *differential of a regular function.* Let $f \in \mathcal{O}(X)$ be a regular function on $X$. We want to define the differential of $f$ at $P$. Since $X$ is closed in $\mathbb{A}^n$, $f$ is induced by a polynomial $F \in K[x_1, \dots, x_n]$ as well as by all polynomials of the form $F + G$ with $G \in I(X)$. Fix $P \in X$: then $d_P(F + G) = d_P F + d_P G$ so the differentials of two polynomials inducing the same function $f$ on $X$ differ by the term $d_PG$ with $G \in I(X)$. By definition, $d_PG$ is zero along $T_{P,X}$, so we may define $d_pf$ as a regular function on $T_{P,X}$, the differential of $f$ at $P$: it is the function on $T_{P,X}$ induced by $d_PF$. Since $d_PF$ is a linear combination of $x_1-y_1,\dots,x_n-y_n$, $d_pf$ can also be seen as an element of $T^*_{P,X}$.

There is a natural map $d_p : \mathcal{O}(X) \to T^*_{P,X}$, which sends $f$ to $d_pf$. Because of the rules of derivation, it is clear that $d_P(f+g) = d_Pf + d_Pg$ and $d_P(fg) = f(P)d_Pg + g(P)d_Pf$. In particular, if $c \in K$, $d_p(cf) = cd_Pf$. So $d_p$ is a linear map of $K$-vector spaces. We denote again by $d_P$ the restriction of $d_P$ to $I_X(P)$, the maximal ideal of the regular functions on $X$ which are zero at $P$. Since clearly $f = f(P) + (f - f(P))$ then $d_Pf = d_P(f - f(P))$, so this restriction doesn't modify the image of the map.

**Proposition 1.3.** *The map $d_P : I_X(P) \to T^*_{P,X}$ is surjective and its kernel is $I_X(P)^2$. Therefore $T^*_{P,X} \simeq I_X(P)/I_X(P)^2$ as $K$-vector spaces.*

*Proof.* Let $\varphi \in T^*_{P,X}$ be a linear form on $T_{P,X}$. $\varphi$ is the restriction of a linear form on $K^n$: $\lambda_1(x_1-y_1)+\ldots+\lambda_n(x_n-y_n)$, with $\lambda_1,\dots,\lambda_n \in K$. Let $G$ be the polynomial of degree 1 $\lambda_1(x_1-y_1)+\ldots+\lambda_n(x_n-y_n)$: the function $g$ induced by $G$ on $X$ is zero at $P$ and coincides with its own differential, so $d_p$ is surjective.

Let now $g \in I_X(P)$ such that $d_pg = 0$, $g$ induced by a polynomial $G$. Note that $d_PG$ may be interpreted as a linear form on $K^n$ which vanishes on $T_{P,X}$, hence as an element of $T^{\perp}_{P,X}$. So $d_PG = c_1d_pF_1 + \ldots + c_md_pF_m$ ($c_1,\dots,c_m$ suitable elements of $K$). Let us consider the polynomial $G - c_1F_1 - \ldots - c_mF_m$: since its differential at $P$ is zero, it doesn't have any term of degree 0 or 1 in $x_1-y_1,\dots,x_n-y_n$, so it belongs to $I(P)^2$. Since $G - c_1F_1 - \ldots - c_mF_m$ defines the function $g$ on $X$, we conclude that $g \in I_X(P)^2$. □

**Corollary 1.4.** *The tangent space $T_{P,X}$ is isomorphic to $(I_X(P)/I_X(P)^2)^*$ as an abstract $K$-vector space.*

**Corollary 1.5.** *Let $\varphi : X \to Y$ be an isomorphism of affine varieties and $P \in X$, $Q = \varphi(P)$. Then the tangent spaces $T_{P,X}$ and $T_{Q,Y}$ are isomorphic.*

*Proof.* $\varphi$ induces the comorphism $\varphi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$, which results to be an isomorphism such that $\varphi^*I_Y(Q) = I_X(P)$ and $\varphi^*I_Y(Q)^2 = I_X(P)^2$. So there is an induced homomorphism

$$I_Y(Q)/I_Y(Q)^2 \to I_X(P)/I_X(P)^2.$$

which is an isomorphism of $K$-vector spaces. By dualizing we get the claim. □

The above map from $T_{P,X}$ to $T_{Q,Y}$ is called the *differential of $\varphi$ at $P$* and is denoted by $d_P\varphi$.

Now we would like to find a "more local" characterization of $T_{P,X}$. To this end we consider the local ring of $P$ in $X$: $\mathcal{O}_{P,X}$. We recall the natural map $\mathcal{O}(X) \to \mathcal{O}_{P,X} = \mathcal{O}(X)_{I_X(P)}$, the last one being the localization. It is natural to extend the map $d_P : \mathcal{O}(X) \to T^*_{P,X}$ to $\mathcal{O}_{P,X}$ setting

$$d_P\Big(\frac{f}{g}\Big) = \frac{g(P)d_Pf - f(P)d_Pg}{g(P)^2}.$$

As in the proof of Proposition 1.3 one proves that the map $d_P : \mathcal{O}_{P,X} \to T^*_{P,X}$ induces an isomorphism $\mathcal{M}_{P,X}/\mathcal{M}^2_{P,X} \to T^*_{P,X}$, where $\mathcal{M}_{P,X}$ is the maximal ideal of $\mathcal{O}_{P,X}$. So by duality we have: $T_{P,X} \simeq (\mathcal{M}_{P,X}/\mathcal{M}^2_{P,X})^*$. This proves that the tangent space $T_{P,X}$ is a *local invariant* of $P$ in $X$.

**Definition 1.6.** Let $X$ be any quasi-projective variety, $P \in X$. The *Zariski tangent space* of $X$ at $P$ is the vector space $(\mathcal{M}_{P,X}/\mathcal{M}^2_{P,X})^*$.

It is an abstract vector space, but if $X \subset \mathbb{A}^n$ is closed, taking the dual of the comorphism associated to the inclusion morphism $X \hookrightarrow \mathbb{A}^n$, we have an embedding of $T_{P,X}$ into $T_{P,\mathbb{A}^n} = \mathbb{A}^n$. If $X \subset \mathbb{P}^n$ and $P \in U_i = \mathbb{A}^n$, then $T_{P,X} \subset U_i$: its projective closure $\mathbb{T}_{P,X}$ is called the *embedded tangent space* to $X$ at $P$.

As we have seen the tangent space $T_{P,X}$ is invariant by isomorphism. In particular its dimension is invariant. If $X \subset \mathbb{A}^n$ is closed, $I(X) = (F_1, \dots, F_m)$, then $\dim T_{P,X} = n - r$, where $r$ is the dimension of the $K$-vector space generated by $\{d_PF_1, \dots, d_pF_m\}$.

Since $d_PF_i = \frac{\partial F_i}{\partial x_1}(P)(x_1 - y_1) + \ldots + \frac{\partial F_i}{\partial x_n}(P)(x_n - y_n)$, $r$ is the rank of the following $m \times n$ matrix, the *Jacobian matrix of $X$ at $P$*:

$$J(P) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(P) & \dots & \frac{\partial F_1}{\partial x_n}(P) \\ \dots & \dots & \dots \\ \frac{\partial F_m}{\partial x_1}(P) & \dots & \frac{\partial F_m}{\partial x_n}(P) \end{pmatrix}.$$

The *generic Jacobian matrix of $X$* is instead the following matrix with entries in $\mathcal{O}(X)$:

$$J = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \dots & \frac{\partial F_1}{\partial x_n} \\ \dots & \dots & \dots \\ \frac{\partial F_m}{\partial x_1} & \dots & \frac{\partial F_m}{\partial x_n} \end{pmatrix}.$$

The rank of $J$ is $\rho$ when all minors of order $\rho + 1$ are functions identically zero on $X$, while at least one minor of order $\rho$ is different from zero at some point. Hence, for all $P \in X$ rk $J(P) \le \rho$, and rk $J(P) < \rho$ if and only if all minors of order $\rho$ of $J$ vanish at $P$. It is then clear that there is a non-empty open subset of $X$ where $\dim T_{P,X}$ is minimal, equal to $n-\rho$, and a proper (possibly empty) closed subset formed by the points $P$ such that $\dim T_{P,X} > n-\rho$.

**Definition 1.7.** The points of an irreducible variety $X$ for which $\dim T_{P,X} = n-\rho$ (the minimal) are called *smooth* or *non-singular* (or *simple) points* of $X$. The remaining points are called *singular* (or multiple). $X$ is a *smooth* variety if all its points are smooth.

If $X$ is quasi-projective, the same argument may be repeated for any affine open subset.

**Example 1.8.** *Let $X \subset \mathbb{A}^n$ be the irreducible hypersurface $V(F)$. Then $J = (\frac{\partial F}{\partial x_1} \dots \frac{\partial F}{\partial x_n})$ is a row matrix. So rk $J = 0$ or $1$. If rk $J = 0$, then $\frac{\partial F}{\partial x_i} = 0$ in $\mathcal{O}(X)$ for all $i$. So $\frac{\partial F}{\partial x_i} \in I(Y) = (F)$. Since the degree of $\frac{\partial F}{\partial x_i}$ is $\leq \deg F - 1$, it follows that $\frac{\partial F}{\partial x_i} = 0$ in the polynomial ring. If the characteristic of $K$ is zero this means that $F$ is constant: a contradiction. If char $K = p$, then $F \in K[x_1^p, \dots, x_n^p]$; since $K$ is algebraically closed, then all coefficients of $F$ are $p$-th powers, so $F = G^p$ for a suitable polynomial $G$; but again this is impossible because $F$ is irreducible. So always rk $J = 1 = \rho$. Hence for $P$ general in $X$, i.e. for $P$ varying in a suitable non-empty open subset of $X$, $\dim T_{P,X} = n-1$. For some particular points, the singular points of $X$, we can have $\dim T_{P,X} = n$, i.e. $T_{P,X} = \mathbb{A}^n$.*

So in the case of a hypersurface $\dim T_{P,X} \geq \dim X$ for every point $P$ in $X$, and equality holds in the points of the smooth locus of $X$. The general case can be reduced to the case of hypersurfaces in view of the following theorem.

**Theorem 1.9.** *Every quasi-projective irreducible variety $X$ is birational to a hypersurface in some affine space.*

*Proof.* We observe that we can reduce the proof to the case in which $X$ is affine, closed in $\mathbb{A}^n$. Let $m = \dim X$. We have to prove that the field of rational functions $K(X)$ is isomorphic to a field of the form $K(t_1, \dots, t_{m+1})$, where $t_1, \dots, t_{m+1}$ satisfy only one non-trivial relation $F(t_1, \dots, t_{m+1}) = 0$, where $F$ is an irreducible polynomial with coefficients in $K$. This will follow from the "Abel's primitive element Theorem" concerning extensions of fields. To state it, we need some preliminaries.

Let $K \subset L$ be an extension of fields. Let $a \in L$ be algebraic over $K$, and let $f_a \in K[x]$ be its minimal polynomial: it is irreducible and monic. Let $E$ be the splitting field of $f_a$.

**Definition 1.10.** An element $a$, algebraic over $K$, is *separable* if $f_a$ does not have any multiple root in $E$, i.e. if $f_a$ and its derivative $f_a'$ don't have any common factor of positive degree. Otherwise $a$ is inseparable. If $K \subset L$ is an algebraic extension of fields, it is called separable if any element of $L$ is separable.

In view of the fact that $f_a$ is irreducible, and that the GCD of two polynomials is independent of the field where one considers the coefficients, if $a$ is inseparable, then $f_a'$ is the zero polynomial. If char $K = 0$, this implies that $f_a$ is constant, which is a contradiction. So in characteristic 0, any algebraic extension is separable. If char $K = p > 0$, then $f_a \in K[x^p]$, and $f_a$ is called an inseparable polynomial. In particular algebraic inseparable elements can exist only in positive characteristic. On the other hand, if the characteristic of $K$ is $p > 0$ and $K$ is algebraically closed, if $f_a = a_0 + a_1x^p + a_2x^{2p} + \cdots + a_kx^{kp}$, then all coefficients are $p$-th powers in $K$, i.e. $a_i = b_i^p$ for suitable elements $b_i$; therefore $f_a = b_0^p + b_1^px^p + b_2^px^{2p} + \cdots + b_k^px^{kp} = (b_0 + b_1x + b_2x^2 + \cdots + b_kx^k)^p$, and this contradicts the irreducibility of $f_a$. We conclude that, if $K$ is algebraically closed, then any algebraic extension is separable.

**Theorem 1.11** (Abel's primitive element Theorem.)**.** *Let $K \subseteq L = K(y_1, \ldots, y_m)$ be an algebraic finite extension. If $L$ is a separable extension, then there exists $\alpha \in L$, called a primitive element of $L$, such that $L = K(\alpha)$ is a simple extension.*

For a proof, see for instance [Lang, Algebra], or any book of Galois theory.

We can now prove Theorem 1.9. The field of rational functions of $X$ is of the form $K(X) = Q(K[X]) = K(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are the coordinate functions on $X$ and $tr.d.K(X)/K = m$. Possibly after renumbering them, we can assume that the first $m$ coordinate functions $t_1, \ldots, t_m$ are algebraically independent over $K$, and $K(X)$ is an algebraic extension of $L := K(t_1, \ldots, t_m)$. So in our situation we can apply Theorem 1.11: there exists a primitive element $\alpha$ such that $K(X) = L(\alpha) = K(t_1 \ldots, t_m, \alpha)$. So there exists an irreducible polynomial $f \in L[x]$ such that $K(X) = L[x]/(f)$. Multiplying $f$ by a suitable element of $K[t_1, \ldots, t_m]$, invertible in $L$, we can eliminate the denominator of $f$ and replace $f$ by a polynomial $g \in K[t_1, \ldots, t_m, x] \subset L[x]$. Now $K[t_1, \ldots, t_m, x]/(g)$ is contained in $L[x](g) = K(X)$, and its quotient field is again $K(X)$. But $K[t_1, \ldots, t_m, x]/(g)$ is the coordinate ring of the hypersurface $Y \subset \mathbb{A}^{m+1}$ of equation $g = 0$. It is clear that $X$ and $Y$ are birationally equivalent, because they have the same fiekd of rational functions. This concludes the proof.

One can show that the coordinate functions on $Y$, $t_1, \ldots, t_{m+1}$, can be chosen to be linear combinations of the original coordinate functions on $X$: this means that $Y$ is obtained as a suitable birational projection of $X$.

**Theorem 1.12.** *The dimension of the tangent space at a non-singular point of an irreducible variety $X$ is equal to* $\dim X$.

*Proof.* It is enough to prove the claim under the assumption that $X$ is affine. Let $Y$ be an affine hypersurface birational to $X$ (which exists by the previous theorem) and $\varphi : X \dashrightarrow Y$

be a birational map. There exist open non-empty subsets $U \subset X$ and $V \subset Y$ such that $\varphi : U \to Y$ is an isomorphism. The set of smooth points of $Y$ is an open subset $W$ of $Y$ such that $W \cap V$ is non-empty and $\dim T_{P,Y} = \dim Y = \dim X$ for all $P \in W \cap V$. But $\varphi^{-1}(W \cap V) \subset U$ is open non-empty and $\dim T_{Q,X} = \dim X$ for all $Q \in \varphi^{-1}(W \cap V)$. This proves the theorem. □

We will denote by $X_{sing}$ the closed set, possibly empty, of singular points of $X$, and by $X_{sm}$ the smooth locus of $X$, i.e. the open non empty subset of its smooth points.

**Corollary 1.13.** *The singular points of an affine variety $X$ closed in $\mathbb{A}^n$ with $\dim X = m$, are the points $P$ of $X$ where the Jacobian matrix $J(P)$ has rank strictly less than $n - m$.*

To find the singular points of a projective variety, it is useful to remember the following Euler relation for homogeneous polynomials.

**Proposition 1.14** (Euler's formula). *Let $F(x_0, \dots, x_n)$ be a homogeneous polynomial of degree $d$. Then $dF = x_0 F_{x_0} + \cdots + x_n F_{x_n}$, where, for every $i = 0, \dots, n$, $F_{x_i}$ denotes the (formal) partial derivative of $F$ with respect to $x_i$.*

*Proof.* Since $d = degF$, we have $F(tx_0, \dots, tx_n) = t^d F(x_0, \dots, x_n)$. To get the desired formula it is enough to derive with respect to $t$ and then put $t = 1$. □

Let now $X \subset \mathbb{P}^n$ be a hypersurface with $I_h(X) = \langle F(x_0, \dots, x_n) \rangle$. Then we have:

**Proposition 1.15.** *The singular points of $X$ are the common zeroes of the partial derivatives of $F$, i.e. $X_{sing} = V_P(F_{x_0}, \dots, F_{x_n})$.*

*Proof.* We denote by $f(x_1, \dots, x_n)$ the dehomogenized ${}^aF = F(1, x_1, \dots, x_n)$ of $F$ with respect to $x_0$. We observe that, for $i = 1, \dots, n$, ${}^a(F_{x_i}) = f_{x_i}$, and that ${}^aF_{x_0} = df - x_1 f_{x_1} - \cdots - x_n f_{x_n}$, in view of Proposition 1.14. So, if $P \in U_0$, $f(P) = f_{x_1}(P) = \cdots = f_{x_n}(P) = 0$ if and only if $F_{x_0}(P) = \cdots = F_{x_n}(P) = 0$. □

Therefore, to look for the singular points of an affine hypersurface $X$, one has to consider the system of equations defined by the equation of $X$ and its partial derivatives, whereas in the projective case it is enough to consider the system of the partial derivatives, because Euler's relation garantees that by consequence also the equation of the hypersurface is satisfied.

For an affine variety $X$ of higher codimension $n - m$, one has to impose the vanishing of the equation of $X$ and of the minors of order $n - m$ of the Jacobian matrix. In the projective case, using again Euler's relation, one can check that the singular points are those that annihilate the homogeneous polynomials $F_1, \dots, F_r$ generating $I_h(X)$ and also the minors of order $n - m$ of the homogeneous $r \times (n+1)$ Jacobian matrix $(\partial F_i / \partial x_j)_{ij}$.

Euler formula is useful also to write the equations of the embedded tangent space $\mathbb{T}_{P,X}$ to a projective variety $X$ at a point $P$. Assume first that $X \subset \mathbb{P}^n$ is a hypersurface $V_P(F)$, $F \in K[x_0, \dots, x_n]$. Assume that $P \in U_0$, and use non-homogeneous coordinates $u_i = x_i/x_0$ on $U_0$, so that $X \cap U_0$ is the zero locus of ${}^aF = F(1, u_1, \dots, u_n) =: f(u_1, \dots, u_n)$. If $P$ has non-homogeneous coordinates $a_1, \dots, a_n$, the affine tangent space $T_{P,X\cap U_0}$ has equation $\sum_{i=1}^n \frac{\partial f}{\partial u_i}(P)(u_i - a_i) = 0$. By definition $\mathbb{T}_{P,X}$ is its projective closure, so it is

$$\{[x_0 \dots, x_n] \mid \sum_{i=1}^n \frac{\partial F}{\partial x_i}(1, a_1, \dots, a_n)(x_i - a_i x_0) = 0\}.$$

From Euler formula, using that $F(1, a_1, \dots, a_n) = 0$, we get that

$$\sum_{i=1}^n \frac{\partial F}{\partial x_i}(1, a_1, \dots, a_n)(-a_i x_0) = \frac{\partial F}{\partial x_0}(1, a_1, \dots, a_n)x_0.$$

We conclude that $\mathbb{T}_{P,X}$ is defined by the equation $\sum_{i=0}^n \frac{\partial F}{\partial x_i}(P)x_i = 0$.

If $X$ is the projective variety with ideal $I_h(X) = (F_1, \dots, F_r)$, then, repeating the previous argument, we get that its tangent space is defined by the linear polynomials $\sum_{i=0}^n \frac{\partial F_k}{\partial x_i}(P)x_i$, for $k = 1, \dots, r$.

We note that the affine tangent space, when $X$ is affine, or the embedded tangent space, when $X$ is projective, to $X$ at $P$ is the intersection of the tangent spaces to the hypersurfaces containing $X$.

Now we would like to study a variety $X$ in a neighbourhood of a smooth point. We have seen that $P$ is smooth for $X$ if and only if $\dim T_{P,X} = \dim X$. Assume $X$ affine: in this case the local ring of $P$ in $X$ is $\mathcal{O}_{P,X} \simeq \mathcal{O}(X)_{I_X(P)}$. But by Theorem 1.8, Lesson 8, we have: $\dim \mathcal{O}_{P,X} = \mathrm{ht}\mathcal{M}_{P,X} = \mathrm{ht}I_X(P) = \dim \mathcal{O}(X) = \dim X$ and $\dim T_{P,X} = \dim_K \mathcal{M}_{P,X}/\mathcal{M}^2_{P,X}$. Therefore $P$ is smooth if and only if

$$\dim_K \mathcal{M}_{P,X}/\mathcal{M}^2_{P,X} = \dim \mathcal{O}_{P,X}$$

(the first one is a dimension as $K$-vector space, the second one is a Krull dimension). By the Nakayama's Lemma a basis of $\mathcal{M}_{P,X}/\mathcal{M}^2_{P,X}$ corresponds bijectively to a minimal system of generators of the ideal $\mathcal{M}_{P,X}$ (observe that the residue field of $\mathcal{O}_{P,X}$ is $K$). Therefore $P$ is smooth for $X$ if and only if $\mathcal{M}_{P,X}$ is minimally generated by $r$ elements, where $r = \dim X$, in other words if and only if $\mathcal{O}_{P,X}$ is a *regular local ring.*

For example, if $X$ is a curve, $P$ is smooth if and only if $T_{P,X}$ has dimension 1, i.e. $\mathcal{M}_{P,X}$ is principal: $\mathcal{M}_{P,X} = (t)$. This means that the equation $t = 0$ only defines the point $P$, i.e. $P$ has one local equation in a suitable neighborhood of $P$.

Let $P$ be a smooth point of $X$ and $\dim X = n$. Functions $u_1, \dots, u_n \in \mathcal{O}_{P,X}$ are called *local parameters* at $P$ if $u_1, \dots, u_n \in \mathcal{M}_{P,X}$ and their residues $\bar{u}_1, \dots, \bar{u}_n$ in $\mathcal{M}_{P,X}/\mathcal{M}^2_{P,X}$ $(= T^*_{P,X})$ form a basis, or equivalently if $u_1, \dots, u_n$ is a minimal set of generators of $\mathcal{M}_{P,X}$. Recalling the isomorphism

$$d_P : \mathcal{M}_{P,X}/\mathcal{M}^2_{P,X} \to T^*_{P,X}$$

we deduce that $u_1, \dots, u_n$ are local parameters if and only if $d_P\bar{u}_1, \dots, d_P\bar{u}_n$ are linearly independent forms on $T_{P,X}$ (which is a vector space of dimension $n$), if and only if the system of equations on $T_{P,X}$

$$d_P\bar{u}_1 = \ldots = d_P\bar{u}_n = 0$$

has only the trivial solution $P$ (which is the origin of the vector space $T_{P,X}$.

Let $u_1, \dots, u_n$ be local parameters at $P$. There exists an open affine neighborhood of $P$ on which $u_1, \dots, u_n$ are all regular. We replace $X$ by this neighborhood, so we assume that $X$ is affine and that $u_1, \dots, u_n$ are polynomial functions on $X$. Let $X_i$ be the closed subset $V(u_i)$ of $X$: it has codimension 1 in $X$, because $u_i$ is not identically zero on $X$ ($u_1, \dots, u_n$ is a minimal set of generators of $\mathcal{M}_{P,X}$).

**Proposition 1.16.** *In this notation, $P$ is a smooth point of $X_i$, for all $i = 1, \dots, n$, and $\bigcap_i T_{P,X_i} = \{P\}$.*

*Proof.* Assume that $U_i$ is a polynomial inducing $u_i$, then $X_i = V(U_i) \cap X = V(I(X) + (U_i))$. So $I(X_i) \supset I(X) + (U_i)$. By considering the linear parts of the polynomials of the previous ideal, we get: $T_{P,X_i} \subset T_{P,X} \cap V(d_PU_i)$. By the assumption on the $u_i$, it follows that $T_{P,X} \cap V(d_PU_1) \cap \dots \cap V(d_PU_n) = \{P\}$. Since $\dim T_{P,X} = n$, we can deduce that $T_{P,X} \cap V(d_PU_i)$ is strictly contained in $T_{P,X}$, and $\dim T_{P,X} \cap V(d_PU_i) = n - 1$. So $\dim T_{P,X_i} \leq n - 1 = \dim X_i$, hence $P$ is a smooth point on $X_i$, equality holds and $T_{P,X_i} = T_{P,X} \cap V(d_PU_i)$. Moreover $\bigcap T_{P,X_i} = \{P\}$. □

Note that $\bigcap_i X_i$ has no positive-dimensional component $Y$ passing through $P$: otherwise the tangent space to $Y$ at $P$ would be contained in $T_{P,X_i}$ for all $i$, against the fact that $\bigcap T_{P,X_i} = \{P\}$.

**Definition 1.17.** Let $X$ be a smooth variety. Subvarieties $Y_1, \dots, Y_r$ of $X$ are called *transversal at $P$*, with $P \in \bigcap Y_i$, if the intersection of the tangent spaces $T_{P,Y_i}$ has dimension as small as possible, i.e. if $\mathrm{codim}_{T_{P,X}}(\bigcap T_{P,Y_i}) = \sum \mathrm{codim}_X Y_i$.

Taking $T_{P,X}$ as ambient variety, one gets the relation:

$$\dim \bigcap T_{P,Y_i} \geq \sum \dim T_{P,Y_i} - (r-1) \dim T_{P,X};$$

hence

$$\operatorname{codim}_{T_{P,X}}(\bigcap T_{P,Y_i}) = \dim T_{P,X} - \dim \bigcap T_{P,Y_i} \leq \sum(\dim T_{P,X} - \dim T_{P,Y_i}) =$$

$$= \sum \operatorname{codim}_{T_{P,X}}(T_{P,Y_i}) \leq \sum \operatorname{codim}_X Y_i.$$

If equality holds, $P$ is a smooth point for $Y_i$ for all $i$, moreover we get that $P$ is a smooth point for the set $\bigcap Y_i$.

For example, if $X$ is a surface and $P \in X$ is smooth, there is a neighbourhood $U$ of $P$ such that $P$ is the transversal intersection of two curves in $U$, corresponding to local parameters $u_1, u_2$. If $P$ is singular we need three functions $u_1, u_2, u_3$ to generate the maximal ideal $\mathcal{M}_{P,X}$.

To conclude this lesson I want to mention the tangent cone to a variety $X$ at a point P. To introduce it we consider first the case where $X$ is a closed affine variety $X \subset \mathbb{A}^n$ and $P = O(0, \dots, 0)$. The tangent cone to $X$ at $O$, $TC_{O,X}$, is the union of the lines through $O$ which are limit positions of secant lines to $X$. To formalize this idea, we consider in $\mathbb{A}^{n+1} = \mathbb{A}^n \times \mathbb{A}^1$ the closed set $\tilde{X}$ of pairs $(a, t)$, with $a = (a_1, \dots, a_n) \in \mathbb{A}^n$ and $t \in \mathbb{A}^1$, such that $at \in X$. Let $\varphi : \tilde{X} \to \mathbb{A}^1$, $\psi : \tilde{X} \to \mathbb{A}^n$ be the projections. If $X \neq \mathbb{A}^n$, $\tilde{X}$ results to be reducible: $\tilde{X} = \tilde{X}_1 \cup \tilde{X}_2$, where $\tilde{X}_2 = \{(a, 0) \mid a \in \mathbb{A}^n\}$, $\tilde{X}_1 = \overline{\varphi^{-1}(\mathbb{A}^1 \setminus 0)}$. We consider the restrictions $\varphi_1, \psi_1$ of the projections to $\tilde{X}_1$. $\psi_1(\tilde{X}_1)$ results to be the closure of the union of the secant lines of $X$ through $O$. The tangent cone $TC_{O,X}$ is by definition $\psi_1(\varphi_1^{-1}(0))$.

Let us write the equations of $TC_{O,X}$. We note first that the equations of $\tilde{X}$ are of the form $F(a_1 t, \dots, a_n t) = 0$ where $F \in I(X)$. Write $F$ as sum of its homogeneous components $F = F_k + \dots + F_d$, where $F_k$ is the non-zero component of minimal degree, and $k \geq 1$ because $O \in X$. Then $F(at) = t^k F(a) + \dots + t^d F_d(a)$. The equation of the component $\tilde{X}_2$ inside $\tilde{X}$ is $t = 0$. The equations of the tangent cones are $F_k = 0$ for all $F \in I(X)$, they are given by the initial forms of the polynomials of $I(X)$. Since all are homogeneous equations, it is clear that we get a cone. Moreover $TC_{O,X} \subseteq T_{O,X}$, and equality holds if and only if $O$ is a smooth point of $X$.

As in the case of the tangent space, we can extend the definition to any point, by translation, and then find a characterization that allows to prove that the tangent cone is invariant by isomorphism.

In the particular case $n = 2$, with $X$ a curve defined by the equation $F(x, y) = 0$, the tangent cone at $O$ is defined by the vanishing of the initial form $F_k(x, y)$. Being a homogeneous polynomial in two variables, it factorizes as a product of $k$ linear forms (counting multiplicities), defining $k$ lines: the tangent lines to $X$ at $O$.

For instance, in the case of the cuspidal cubic $V(x^3 - y^2)$ the tangent cone has equation $y^2 = 0$: it is the line $y = 0$ "counted with multiplicity 2. If $X$ is the cubic of equation

$x^2 - y^2 + x^3 = 0$, the tangent cone consists in the two distinct lines $x - y = 0$ and $x + y = 0$: the cubic is nodal.

The tangent cone allows to define the multiplicity of a point on $X$ and to start an analysis of the singularities.

**Exercises 1.18.** 1. Assume char $K \neq 2$. Find the singular points of the following surfaces in $\mathbb{A}^3$:

(1) $xy^2 = z^3$;
(2) $x^2 + y^2 = z^2$;
(3) $xy + x^3 + y^3 = 0$.

2. Suppose that char $K \neq 3$. Determine the singular locus of the projective variety in $\mathbb{P}^5$ given by the equations:

$$\sum_{i=0}^{5} x_i = 0, \quad \sum_{i=0}^{5} x_i^3 = 0.$$