Exercise - Yukawa theory.

Consider the theory of a real scalar field ϕ and a fermion ψ in $d = 4$ spacetime dimensions with Lagrangian

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 + \bar{\psi}(i\partial\!\!\!/ - M)\psi + y\phi\bar{\psi}\psi \,. \tag{1}$$

- For $m > 2M$ compute the tree-level decay rate for $\phi \to \bar{\psi}\psi$.

The scalar propagator can be written as $iG(p^2) = i(p^2 - m^2 + \Sigma(p^2) + i\varepsilon)^{-1}$, where $i\Sigma$ is the 1PI contribution to the two-point function. The counter-terms, necessary to absorb the UV divergencies, contribute as $i\Sigma_{\text{ct}} = i(p^2\delta_\phi - (\delta_m + \delta_\phi)m^2)$.

- Compute $i\Sigma(p^2)$ at one-loop, regularising the theory in dim-reg, fixing the counter-terms in the $\overline{\text{MS}}$ scheme.
- For $m > 2M$ compute the imaginary part of the propagator and check the optical theorem, comparing with the decay rate.
- Compute the difference between the pole and renormalised $\overline{\text{MS}}$ mass

$$\Delta m^2 \equiv m_P^2 - m^{\overline{\text{MS}}}(\mu) = f(m_P, M, \mu). \tag{2}$$

- What happens in the limit $M \gg m_P$ to the ratio $\Delta m^2/m_P^2$? How can you interpret this result?