LESSON 18.

1. Finite morphisms and blow–ups.

In this section we will see the notion of finite morphism, and a fundamental example of a morphism which is not finite: the blow-up of a variety at a point, or, more in general, along a subvariety. The blow-up is the main ingredient in the resolution of singularities of an algebraic variety. As usual we will assume that K is algebraically closed.

First of all we will give an interpretation in geometric terms of the notions of integral elements and integral extensions introduced and studied in Lessons 5 and 9.

Let $f : X \to Y$ be a dominant morphism of affine varieties, i.e. we assume that $f(X)$ is dense in Y. Then the comorphism $f^* : K[Y] \to K[X]$ is injective (by Ex. 4, Lesson 13): we will identify $K[Y]$ with its image $f^*K[Y] \subset K[X]$.

Definition 1.1. f is a finite morphism if $K[X]$ is an integral extension of $K[Y]$.

This means that, for any regular function φ on X, there is a relation of integral dependence

$$\varphi^r + f^*(g_1)\varphi^{r-1} + \cdots + f^*(g_r) = 0 \qquad (1)$$

with $g_1, \ldots, g_r \in K[Y]$. Finite morphisms enjoy the following properties.

Proposition 1.2. (1) *The composition of finite morphisms is a finite morphism.*

(2) *Let $f : X \to Y$ be a finite morphism of affine varieties. Then, for any $y \in Y$, $f^{-1}(y)$ is a finite set.*

(3) *Finite morphisms are surjective, i.e. $f^{-1}(y)$ is non-empty for any $y \in Y$.*

(4) *Finite morphisms are closed maps.*

Proof. (1) It follows from the transitivity of integral dependence, Lesson 5, Corollary 1.2.

(2) Let X be a closed subset of $\mathbb{A}^n$, so $K[X]$ is generated by the coordinate functions $t_1, \ldots, t_n$. Let $y \in Y$. We want to prove that any coordinate function t_i takes only a finite number of values on the set $f^{-1}(y)$. For the function t_i there is a relation of integral dependence of type (1): $t_i^r + f^*(g_1)t_i^{r-1} + \cdots + f^*(g_r) = 0 \in K[X]$ with $g_1, \ldots, g_r \in K[Y]$. We apply this relation to $x \in f^{-1}(y)$ and we get $t_i^r(x) + g_1(y)t_i^{r-1}(x) + \cdots + g_r(y) = 0$. This means that the i-th coordinate of any point in $f^{-1}(y)$ has to satisfy an equation of degree r, so there are only finitely many possibilities for this coordinate. This proves what we want.

(3) This is a consequence of the property of Lying over - LO (Lesson 9, Theorem 1.3). Let $y = (y_1, \dots, y_m) \in Y \subset \mathbb{A}^m$, let $u_1, \dots, u_m$ be the coordinate functions on Y. A point $x \in X$ belongs to $f^{-1}(y)$ if and only if $u_i(f(x)) = f^*(u_i)(x) = y_i$ for any i, or equivalently if and only if the function $f^*(u_i) - y_i$ vanishes on x. In view of the relative version of the Nullstellensatz, the condition $f^{-1}(y) = \emptyset$ is therefore equivalent to the fact that the ideal generated by $f^*(u_1) - y_1, \dots, f^*(u_m) - y_m$ in $K[X]$ is the entire ring $K[X]$. Consider now the maximal ideal $I_Y(y)$ of regular functions on Y vanishing in y, it is generated by $u_1 - y_1, \dots, u_m - y_m$. From the Lying over applied to the integral extension $f^*K[Y] \subset K[X]$, it follows that there is a prime ideal $\mathcal{P}$ of $K[X]$ over $f^*(I_Y(y))$, which is generated by $f^*(u_1) - y_1, \dots, f^*(u_m) - y_m$. This implies that $f^{-1}(y) \neq \emptyset$.

(4) Let $f : X \to Y$ be a finite morphism and $Z \subset X$ an irreducible closed subset. We consider the restriction of f to Z, i.e. $\bar{f} : Z \to \overline{f(Z)}$. We observe that, via the comorphism $\bar{f}^* : K[\overline{f(Z)}] \to K[Z]$, $K[Z] \simeq K[X]/I_X(Z)$ is an integral extension of $K[\overline{f(Z)}]$, because it is enough to reduce modulo $I_X(Z)$ the integral equations of the elements of X. So, using (3), we conclude that $\bar{f}$ is surjective, i.e. $f(Z) = \overline{f(Z)}$.

□

An example of non-finite morphism is the projection $V(xy - 1) \to \mathbb{A}^1$. Instead the projection $p_2 : V(y - x^2) \to \mathbb{A}^1$ is finite.

Theorem 1.3 (Geometric interpretation of the Normalisation Lemma)**.** *Let $X \subset \mathbb{A}^n$ be an affine irreducible variety of dimension d. Then there exists a finite morphism $X \to \mathbb{A}^d$. Moreover the morphism can be taken to be a projection.*

Proof. The coordinate ring of X is an integral K-algebra, finitely generated by the coordinate functions, whose quotient field has transcendence degree d over K. The Normalization Lemma (Theorem 1.3, Lesson 5) then asserts that there exist elements $z_1, \dots, z_d$ algebraically independent over K, such that $K[X]$ is an integral extension of the K-algebra $B = K[z_1, \dots, z_d]$. But B is the coordinate ring of $\mathbb{A}^d$ and the inclusion $B \hookrightarrow K[X]$ can be seen as the comorphism of a finite morphism $f : X \to \mathbb{A}^d$. The proof of Normalization Lemma shows that $z_1, \dots, z_d$ can be chosen linear combinations of the generators of $K[X]$. In this case, f results to be a projection. □

One can prove that being a finite morphism is a local property, in the following sense: let $f : X \to Y$ be a morphism of affine varieties. Then f is finite if and only if any $y \in Y$ has an affine open neighbourhood V, such that $U := f^{-1}(V)$ is affine, and the restriction $f\,| : U \to V$ is a finite morphism. This property allows to give the definition of finite morphism between arbitrary varieties, as a morphism which is finite when restricted to the open subsets of an affine open covering. See [Šafarevič] for more details and consequences.

For instance one can obtain the following non-trivial facts, that I quote here only for information.

Example 1.4. *1. Let $X \subset \mathbb{P}^n$ be a closed algebraic set, let $\Lambda \subset \mathbb{P}^n$ be a linear subspace of dimension d such that $X \cap \Lambda = \emptyset$. Then the restriction of the projection $\pi_\Lambda : X \to \mathbb{P}^{n-d-1}$ defines a finite morphism from X to $\pi_\Lambda(X)$.*

2. Let $X \subset \mathbb{P}^n$ be a closed algebraic set and $F_0, \dots, F_r$ be homogeneous polynomials of the same degree d without any common zero on X. Then $\varphi : X \to \mathbb{P}^r$ defined by the polynomials $F_0, \dots, F_r$ is a finite morphism to the image.

For a proof of the first property, see [Šafarevič]. To prove the second one, we observe that φ is the composition of the Veronese morphism $v_{n,d}$ with a projection. The conclusion follows from part 1., remembering that $v_{n,d}$ is an isomorphism.

We will define now the blow-up (or blowing-up) of an affine space at the origin $O(0, \dots, 0)$. It is a variety X with a morphism $\sigma : X \to \mathbb{A}^n$ which results to be birational and not finite. The idea is that X is obtained from $\mathbb{A}^n$ by replacing the point O with a $\mathbb{P}^{n-1}$, which can be interpreted as $\mathbb{P}(T_{O,\mathbb{A}^n})$, the set of the tangent directions to $\mathbb{A}^n$ at O.

To construct X we first consider the product $\mathbb{A}^n \times \mathbb{P}^{n-1}$, which is a quasi-projective variety via the Segre map. Let $x_1, \dots, x_n$ be the coordinates of $\mathbb{A}^n$, and $y_1, \dots, y_n$ the homogeneous coordinates of $\mathbb{P}^{n-1}$. We recall that the closed subsets of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ are zeros of polynomials in the two series of variables, which are homogeneous in $y_1, \dots, y_n$.

Definition 1.5. Let X be the closed subset of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ defined by the system of equations

$$\left\{ x_i y_j = x_j y_i, i, j = 1, \dots, n. \right. \tag{2}$$

The blow-up of $\mathbb{A}^n$ at O is the variety X together with the map $\sigma : X \to \mathbb{A}^n$ defined by restricting the first projection of $\mathbb{A}^n \times \mathbb{P}^{n-1}$. O is also called the centre of the blow-up.

The equations (2) express that $y_1, \dots, y_n$ are proportional to $x_1, \dots, x_n$. Let us see what this means. Let $P \in \mathbb{A}^n$ be a point, we consider $\sigma^{-1}(P)$. We distinguish two cases:

1) If $P \neq O$, then $\sigma^{-1}(P)$ consists of a single point and precisely, if $P = (a_1, \dots, a_n)$, $\sigma^{-1}(P)$ is the pair $((a_1, \dots, a_n), [a_1, \dots, a_n])$.

2) If $P = O$, then $\sigma^{-1}(O) = \{O\} \times \mathbb{P}^{n-1} \simeq \mathbb{P}^{n-1}$, because if $x_1 = \cdots = x_n = 0$ there are no restrictions on $y_1 \dots, y_n$. It is a standard notation to denote $\sigma^{-1}(O)$ by E. It is called the *exceptional divisor* of the blow-up.

It is easy to check that σ gives an isomorphism between $X \setminus \sigma^{-1}(O)$ and $\mathbb{A}^n \setminus \{O\}$. Indeed both σ and σ^{-1} so restricted are regular.

The points of $\sigma^{-1}(O)$ are in bijection with the set of lines through O in $\mathbb{A}^n$. Indeed if L is a line through O, it can be parametrized by $\{x_i = a_i t,\ t \in K$, with $(a_1, \dots, a_n) \neq (0, \dots, 0)$. Then $\sigma^{-1}(L \setminus O)$ is parametrized by

$$\begin{cases} x_i = a_i t \\ y_i = a_i t, t \neq 0, \end{cases} \tag{3}$$

or, which is the same, by

$$\begin{cases} x_i = a_i t \\ y_i = a_i, t \neq 0. \end{cases} \tag{4}$$

If we add also $t = 0$, we find the closure $L' = \overline{\sigma^{-1}(L \setminus O)}$, it is a line meeting $\sigma^{-1}(O)$ at the point $O \times [a_1, \dots, a_n]$: L' can be interpreted as the line L "lifted at the level $[a_1, \dots, a_n]$". So we have a bijection associating to the line L passing through O the point $\overline{\sigma^{-1}(L \setminus O)} \cap \sigma^{-1}(O) = L' \cap E$.

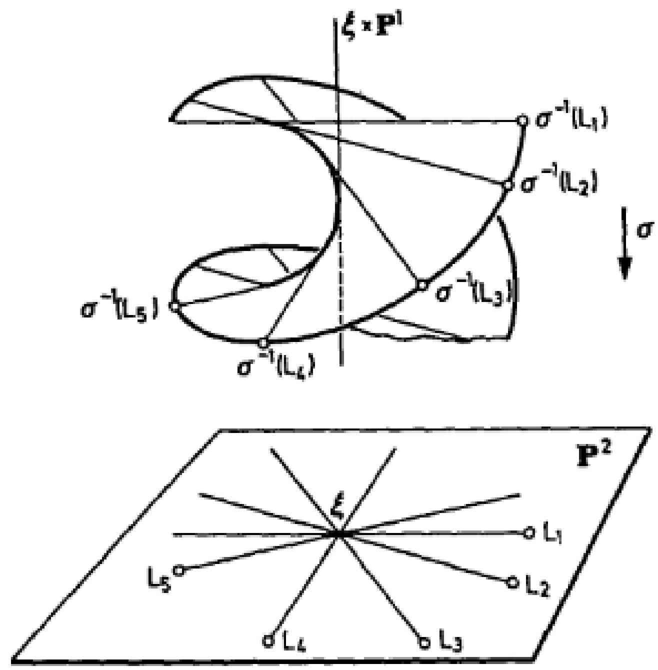

FIGURE 1

Finally we note that X is irreducible: indeed $X = (X \setminus E) \cup E$; $X \setminus E$ is isomorphic to $\mathbb{A}^n \setminus O$, so it is irreducible; moreover every point of E belongs to a line L', the closure of $\sigma^{-1}(L \setminus O) \subset X \setminus E$. Hence $X \setminus E$ is dense in X, which implies that X is irreducible.

Therefore X is birational to $\mathbb{A}^n$: they are both irreducible and contain the isomorphic open subsets $X \setminus \sigma^{-1}(O)$ and $\mathbb{A}^n \setminus O$. In particular $\dim X = n$, and $\sigma^{-1}(O) = E \simeq \mathbb{P}^{n-1}$ has codimension 1 in X. The tangent space $T_{O,\mathbb{A}^n}$ coincides with $\mathbb{A}^n = K^n$, and the set of the lines through O can be interpreted as the projective space $\mathbb{P}(T_{O,\mathbb{A}^n})$. So there is a bijection between the exceptional divisor E and $\mathbb{P}(T_{O,\mathbb{A}^n})$.

Figure 1, taken from the book of Šafarevič, illustrates the case of the plane.

If we consider the second projection $p_2 : X \to \mathbb{P}^{n-1}$, for any $[a] = [a_1, \dots, a_n] \in \mathbb{P}^{n-1}$, $p_2^{-1}[a]$ is the line L' of (4). X with the map p_2 is an example of non-trivial line bundle, called the universal bundle over $\mathbb{P}^{n-1}$.

If Y is a closed subvariety of $\mathbb{A}^n$ passing through O, it is clear that $\sigma^{-1}(Y)$ contains the exceptional divisor $E = \sigma^{-1}(O)$. It is called the total trasform of Y in the blow-up. We define the *strict transform of* Y in the blow-up of $\mathbb{A}^n$ as the closure $\widetilde{Y} := \overline{\sigma^{-1}(Y \setminus O)}$. It is interesting to consider the intersection $\widetilde{Y} \cap E$, it depends on the behaviour of Y in a neighborhood of O, and allows to analyse its singularities at O.

Example 1.6.

1. Let $Y \subset \mathbb{A}^2$ be the plane cubic curve of equation $y^2 - x^2 = x^3$. The origin is a singular point of Y, with multiplicity 2, and the tangent cone $TC_{O,Y}$ is the union of the two lines of equations $x - y = 0$, $x + y = 0$, respectively. We consider the blow-up $X \subset \mathbb{A}^2 \times \mathbb{P}^1$ of $\mathbb{A}^2$ with centre O. Using coordinates t_0, t_1 in $\mathbb{P}^1$, X is defined by the unique equation $xt_1 = t_0 y$. Then $\sigma^{-1}(Y)$ is defined by the system

$$\begin{cases} y^2 - x^2 = x^3 \\ xt_1 = t_0 y \end{cases}$$

As usual $\mathbb{P}^1$ is covered by the two open subsets $U_0 : t_0 \neq 0$ and $U_1 : t_1 \neq 0$, so $\mathbb{A}^2 \times \mathbb{P}^1 = (\mathbb{A}^2 \times U_0) \cup (\mathbb{A}^2 \times U_1)$, the union of two copies of $\mathbb{A}^3$, and we can study X considering its intersection X_0, X_1 with each of them. If $t_0 \neq 0$, we use $t = t_1/t_0$ as affine coordinate; if $t_1 \neq 0$ we use $u = t_0/t_1$. X_0 has equation $y = tx$ and X_1 has equation $x = uy$. For $\sigma^{-1}(Y) \cap X_0$ we get the equations $y^2 - x^2 - x^3 = 0$ and $y = tx$ in $\mathbb{A}^3$ with coordinates x, y, t. Substituting we get $t^2x^2 - x^2 - x^3 = x^2(t^2 - 1 - x) = 0$. So there are two components: one is defined by $x = y = 0$, which is $E \cap X_0$; the other is defined by $\begin{cases} x = t^2 - 1 \\ y = t(t^2 - 1) \end{cases}$, it is $\widetilde{Y} \cap X_0$. Note that it meets E at the two points $P(0, 0, 1), Q(0, 0, -1)$. They correspond on E to the two tangent lines to Y at O: $y - x = 0$ and $x + y = 0$.

If we work on the other open set $\mathbb{A}^2\times U_1$, $\sigma^{-1}(Y)$ is defined by $x = uy$ and $y^2-u^2y^2-u^3y^3 = y^2(1-u^2-u^3y) = 0$. So $\widetilde{Y}\cap X_1$ is defined by $\begin{cases} x = uy \\ 1-u^2-u^3y = 0 \end{cases}$. We find the same two points of intersection with E: $(0,0,1),(0,0,-1)$.

The restriction of the projection $\sigma : \widetilde{Y} \to Y$ is an isomorphism outside the points P, Q on $\widetilde{Y}$ and O on Y. The result is that the two branches of the singularity O have been separated, and the singularity has been resolved.

2. Let $Y \subset \mathbb{A}^2$ be the cuspidal cubic curve of equation $y^2 - x^3 = 0$. The total transform is defined by

$$\begin{cases} y^2 - x^3 = 0 \\ xt_1 = t_0y. \end{cases}$$

On the first open subset it becomes $y^2 - x^3 = 0$ together with $y = tx$; replacing and simplifying t, which corresponds to E, we get the equations for $\widetilde{Y}$:

$$\begin{cases} x = t^2 \\ y = t^3 \end{cases}.$$

This is the affine skew cubic, that meets E at the unique point $(0,0,0)$, corresponding to the tangent line to Y at O: $y = 0$. By the way, we can check that E is the tangent line to $\widetilde{Y}$ at $(0,0,0)$. On the second open subset, we have the equations $y^2 - x^3 = 0$ together with $x = uy$; the strict transform is defined by $1 - u^3y = 0$ and $x = uy$. There is no point of intersection with E in this affine chart. The map $\sigma : \widetilde{Y} \to Y$ is therefore regular, birational, bijective, but not biregular; Y and $\widetilde{Y}$ cannot be isomorphic, because one is smooth and the other is not smooth.

3. Let $Y = V(x^2 - x^4 - y^4) \subset \mathbb{A}^2$. O is a singular point of multiplicity 2 with tangent cone the line $x = 0$ counted twice. Let $\widetilde{Y}$ be the strict transform of Y in the blow-up of the plane in the origin. Proceeding as in the previous example we find that $\widetilde{Y}$ meets the exceptional divisor $E = O \times \mathbb{P}^1$ at the point $O' = ((0,0),[0,1])$, which belongs only to the second open subset $\mathbb{A}^2 \times U_1$. In coordinates $x, y, u = t_0/t_1$, $\widetilde{Y}$ is defined by the equations

$$\begin{cases} x = uy \\ u^2 - u^4y^2 - y^2 = 0 \end{cases},$$

and $O' = (0,0,0)$. We compute the equation of the tangent space $T_{O',\widetilde{Y}}$, it is $x = 0$: it is a 2-plane in $\mathbb{A}^3$, so $\widetilde{Y}$ is singular at O'. The tangent cone $TC_{O',\widetilde{Y}}$ is $x = 0, u^2 - y^2 = 0$, the union of two lines in the tangent plane.

Let us consider a second blow-up σ', of $\mathbb{A}^3$ in O'. It is contained in $\mathbb{A}^3 \times \mathbb{P}^2$; using coordinates z_0, z_1, z_2 in $\mathbb{P}^2$, it is defined by

$$rk \begin{pmatrix} x & y & u \\ z_0 & z_1 & z_2 \end{pmatrix} \leq 2.$$

If we work on the open subset $\mathbb{A}^3 \times U_0 \simeq \mathbb{A}^5$, with coordinates $x, y, u, \zeta_1 = z_1/z_0, \zeta_2 = z_2/z_0$, the exceptional divisor E' is defined by $x = y = u = 0$, and the total transform $\widetilde{Y}'$ of $\widetilde{Y}$ by

$$\begin{cases} x = uy \\ y = \zeta_1 x \\ u = \zeta_2 x \\ \zeta_2^2 x^2 - \zeta_1^2 x^2 (1 + \zeta_2^4 x^4) = 0 \end{cases}.$$

In the last equation there is a common factor x^2, that gives rise to the exceptional divisor; after simplifying it we obtain the equation $\zeta_2^2 - \zeta_1^2(1 + \zeta_2^4 x^4) = 0$, which defines the strict transform together with $x = uy, y = \zeta_1 x, u = \zeta_2 x$. The intersection $\widetilde{Y}' \cap E'$ is given therefore by $x = y = u = \zeta_2^2 - \zeta_1^2 = 0$, two points P, Q. Considering the two other open sets $\mathbb{A}^3 \times U_1$, $\mathbb{A}^3 \times U_2$, we find the same points P, Q.

In conclusion, we consider the composition of the two blow-ups $\widetilde{Y}' \xrightarrow{\sigma'} \widetilde{Y} \xrightarrow{\sigma} Y$, which is birational. In the first blow-up σ, we pass from Y, with a singularity at the blown-up point O with one tangent line, to $\widetilde{Y}$ with a node in O', its point of intersection with E. In the second blow-up σ', O' is replaced by two points on the second exceptional divisor E'. To verify if $\widetilde{Y}'$ is smooth, it is enough to check if P, Q are smooth, and this can be checked easily.

The singularity of Y is called a *tacnode*. We have just checked that to resolve it two blow-ups are needed. What allows to distinguish the singularity of the curve of Example 2 from the present example, is the multiplicity of intersection at the point O of the tangent line at the singular point O with the curve: it is 3 in Example 2 and 4 in Example 3.

The general problem of the *resolution of singularities* is, given a variety Y, to find a birational morphism $f : Y' \to Y$ with Y' non-singular. It is possible to prove that, if Y is a curve, the problem can be solved with a finite sequence of blow-ups. If $\dim Y > 1$, the problem is much more difficult, and is presently completely solved only in characteristic 0 (see for instance [Hartshorne], Ch. V, 3).

To conclude this Lesson, we will see a different way to introduce the blow-up of $\mathbb{A}^n$ at O. Let $\pi : \mathbb{A}^n \setminus O \to \mathbb{P}^{n-1}$ be the natural projection $(a_1, \dots, a_n) \to [a_1, \dots, a_n]$. Let Γ be the graph of π, $\Gamma \subset (\mathbb{A}^n \setminus O) \times \mathbb{P}^{n-1} \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$. We immediately have that the closure of

Γ in $\mathbb{A}^n \times \mathbb{P}^{n-1}$ is precisely the blow-up X of $\mathbb{A}^n$ at O. This interpretation suggests how to extend Definition 1.5 and define the blow up of a variety X along a subvariety Y.

Suppose that X is an affine variety and $I = I_X(Y) \subset K[X]$ is the ideal of a subvariety Y of X. Suppose that $I = (f_0, \dots, f_r)$. Let λ be the rational map $X \dashrightarrow \mathbb{P}^r$ defined by $\lambda = [f_0, \dots, f_r]$. The blow-up of Y is the closure of the graph of λ with the projection to X. Similarly one can define the blow-up of a projective variety along a subvariety defined by an ideal generated by homogeneous polynomials all of the same degree. For details, see for instance [Cutkosky].

Exercises 1.7. Sia $Y \subset \mathbb{P}^2$ be a smooth plane projective curve of degree $d > 1$, defined by the equation $f(x, y, z) = 0$. Let $C(Y) \subset \mathbb{A}^3$ be the affine variety defined by the same polynomial f: $C(Y)$ is the affine cone of Y. Let $O(0, 0, 0) \in \mathbb{A}^3$ be the origin, vertex of $C(Y)$. Let $\sigma : X \to \mathbb{A}^3$ be the blow-up in O.

1. Show that $C(Y)$ has only one singular point, the vertex O;
2. show that $\widetilde{C(Y)}$, the strict transform of $C(Y)$, is nonsingular (cover it with open affine subsets);
3. let E be the exceptional divisor; show that $\widetilde{C(Y)} \cap E$ is isomorphic to Y.