LESSON 20.

1. Fibres of a morphism and lines on hypersurfaces.

In this last Lession, we will state the Theorem on the dimension of the fibres of a morphism, and we will see an application, involving Grassmannians, about the existence of lines on a hypersurface of given degree in a projective space.

Let us recall that the *fibres of a morphism* are the inverse images of the points of the codomain. More precisely, if $f : X \to Y$ is a morphism, for any $y \in Y$, the fibre of f over y is $f^{-1}(y)$. Since in the Zariski topology every point is closed, the fibre $f^{-1}(y)$ is closed in X, and we want to study the dimensions of its irreducible components. We have seen in Lesson 18 that finite morphisms have the property that all the fibres are finite and non-empty, so all irreducible components have dimension 0.

The following theorem gives informations about the behaviour of the fibres of general morphisms.

Theorem 1.1 (Theorem on the dimension of the fibres.). *Let $f : X \to Y$ be a dominant morphism of algebraic sets. Then:*

1. $\dim(X) \geq \dim(Y)$*;*

2. for any $y \in Y$, and for any irreducible component F of $f^{-1}(y)$, $\dim F \geq \dim(X) - \dim(Y)$*;*

3. there exists a non-empty open subset $U \subset Y$, such that $\dim f^{-1}(y) = \dim(X) - \dim(Y)$ *for any $y \in U$;*

4. the sets $Y_k = \{y \in Y \mid \dim f^{-1}(y) \geq k\}$ are closed in Y (upper semicontinuity of the dimension of the fibres).

Before giving a sketch of the proof, let us see an example.

Example 1.2. *Let V be an affine variety and consider $W \subset V \times \mathbb{A}^r$ defined by s linear equations with coefficients in $K[V]$:*

$$\sum_{j=1}^{r} a_{ij}x_j,\ a_{ij} \in K[X],\ i = 1, \dots, s.$$

Let $\varphi : W \to V$ *be the projection. For* $P \in V$, $\varphi^{-1}(P)$ *is the set of solutions of the system of linear equations with constant coefficients*

$$\sum_{j=1}^{r} a_{ij}(P)x_j,\ a_{ij}(P) \in K,\ i = 1, \dots, s,$$

so its dimension is $r - rk(a_{ij}(P))$. *For any* $k \in \mathbb{N}$ *the set* $\{P \in V \mid rk(a_{ij}(P)) \leq k\}$ *is closed in* V, *defined by the vanishing of the minors of order* $r+1$, *and it is precisely* V_{r-k}, *the subset of* V *where the dimension of the fibre is* $\geq r - k$.

Proof of Theorem 1.1. 1. Since f is dominant, there is the K-homomorphism $f^* : K(Y) \hookrightarrow K(X)$, and $tr.d.K(Y)/K \leq tr.d.K(X)/K$, because algebraically independent elements of $K(Y)$ remain algebraically independent in $K(X)$. So $\dim(Y) \leq \dim(X)$.

2. Fix $y \in Y$. We observe that we can replace Y with an affine open neighborhood U of y and X with $f^{-1}(U)$. So we can assume that Y is closed in an affine space $\mathbb{A}^N$. Let $n = \dim(X), m = \dim(Y)$. We observe that we can find a polynomial G in N variables which does not vanish identically on any irreducible component of Y. For instance, we can fix a point on any irreducible component and choose a hyperplane not passing through any of these points. Then all irreducible components of $Y^{(1)} := Y \cap V(G)$ have dimension $m-1$. Repeating this argument, we can find a chain of subvarieties of Y of the form $Y \supset Y^{(1)} \supset \cdots \supset Y^{(m)} \supset Y^{(m+1)}$, where all irreducible components of $Y^{(i)}$ have dimension $m - i$. In particular the irreducible components of $Y^{(m)}$ are points, among which there is y, and $Y^{(m)}$ is defined by m equations of the form $g_1 = \cdots = g_m = 0$, with $g_1, \dots, g_m \in K[Y]$. Possibly restricting the open set U, we can assume that $Y^{(m)} \cap U = \{y\}$. Hence, the fibre $f^{-1}(y)$ is defined by the system of m equations $f^*(g_1) = \cdots = f^*(g_m) = 0$. The conclusion follows from the Theorem of the intersection (Lesson 15, Theorem 1.1).

3. See [Šafarevič].

4. By induction on the dimension of Y. It is obviously true if $\dim Y = 0$. We know from 3. that there is an open subset U of Y such that $\dim f^{-1}(y) = n - m$ if and only if $y \in U$. Let Z be the complement of U in Y; thus $Z = Y_{n-m+1}$. Let $Z_1, \dots, Z_r$ be the irreducible components of Z. We can now apply the induction to the restrictions of f, $f^{-1}(Z_j) \to Z_j$ for each j , and we obtain the result. □

As a consequence of Theorem 1.1, we are able to prove the following very useful proposition.

Proposition 1.3. *Let* $f : X \to Y$ *be a surjective morphism of projective algebraic sets. Assume that* Y *is irreducible and that all fibres of* f *are irreducible and of the same dimension* r, *then* X *is irreducible of dimension* $\dim(Y) + r$.

Proof. Note first of all that $r = \dim(X) - \dim(Y)$. Let Z be an irreducible closed subset of X, and consider the restriction $f|_Z : Z \to Y$; its fibres are $f|_Z^{-1}(y) = f^{-1}(y) \cap Z$. There are three possibilities:

(a) $f(Z) \neq Y$. Then $f(Z)$ is a proper closed subset of Y;

(b) $f(Z) = Y$ and $\dim(Z) < r + \dim(Y)$. Then 2. of Theorem 1.1 shows that there is a nonempty open subset U of Y such that for $y \in U$, $\dim(f^{-1}(y) \cap Z) = \dim(Z) - \dim(Y) < r = \dim(X) - \dim(Y)$. Thus, for $y \in U$, the fibre is not contained in Z.

(c) $f(Z) = Y$ and $\dim(Z) \geq r + \dim(Y)$. Then again 2. of Theorem 1.1 shows that $\dim(f^{-1}(y) \cap Z) \geq \dim(Z) - \dim(Y) \geq r$ for all y; thus $f^{-1}(y) \subset Z$ for all $y \in Y$, so $Z = X$.

Now let $Z_1, \dots, Z_r$ be the irreducible components of X . We claim that (c) holds for at least one of the Z_i. Otherwise, there will be an open subset U in Y, such that for $y \in U$, $f^{-1}(y)$ is contained in none of the Z_i; but $f^{-1}(y)$ is irreducible and $f^{-1}(y) = \bigcup_i (f^{-1}(y) \cap Z_i)$ so this is impossible. We conclude that X is irreducible. □

As an important application, we will study the existence of lines on hypersurfaces of fixed degree. Let $S = K[x_0, \dots, x_n]$, let $d \geq 1$ be an integer number, then $\mathbb{P}(S_d)$ is a projective space of dimension $N = \binom{n+d}{d} - 1$, parametrising the hypersurfaces of degree d in $\mathbb{P}^n$. Among them there are reducible and even non-reduced hypersurfaces (i.e. those corresponding to non square-free polynomials). Let us introduce the *incidence correspondence* line-hypersurface as follows. Let $\mathbb{G}(1, n)$ be the Grassmannian parametrising the lines in $\mathbb{P}^n$. We consider the product variety $\mathbb{G}(1, n) \times \mathbb{P}(S_d)$, whose points are the pairs $(\ell, [F])$, where ℓ is a line in $\mathbb{P}^n$ and $F \in S_d$, that we can identify with the hypersurface $V_P(F)$. By definition the incidence variety (or correspondence) is $\Gamma_d := \{(\ell, [F]) \mid \ell \subset V_P(F)\} \subset \mathbb{G}(1, n) \times \mathbb{P}(S_d)$.

Proposition 1.4. *Γ_d is a projective algebraic set, i.e. it is the set of zeros of a set of bihomogeneous polynomials in two series of variables: the Plücker coordinates p_{ij} on the Grasmannian and the coefficients $a_{i_0 \dots i_n}$ of F.*

Proof. Let $P = (p_{ij})$ be the skew-symmetric matrix, whose elements are the coordinates of a line ℓ: it has rank two and from Proposition 1.8, Lesson 19, it follows that each non-zero row of P contains the coordinates of a point of ℓ. So the rows of P are a system of generators of a vector subspace W of dimension 2, such that $\ell = \mathbb{P}(W)$. Hence the coordinates of any point of ℓ are linear combinations of the rows of P, of the form $(x_0 = \Sigma_i \lambda_i p_{0i}, \dots, x_n = \Sigma_i \lambda_i p_{ni})$. A line ℓ is contained in $V_P(F)$ if and only if the equation $F(\Sigma_i \lambda_i p_{0i}, \dots, \Sigma_i \lambda_i p_{ni}) = 0$ is an identity in $\lambda_0, \dots, \lambda_n$. Therefore, Γ_d is the set of common zeros of the coefficients of the monomials of degree d in $\lambda_0, \dots, \lambda_n$: they are homogeneous of degree 1 in the coefficients of F and of degree d in the p_{ij}'s. □

Example 1.5.

Let $n = d = 3$, $F = x_0^3 - x_1x_2x_3 \in S_3$. We put

$$\begin{cases} x_0 = \lambda_1 p_{01} + \lambda_2 p_{02} + \lambda_3 p_{03} \\ x_1 = -\lambda_0 p_{01} + \lambda_2 p_{12} + \lambda_3 p_{13} \\ x_2 = -\lambda_0 p_{02} - \lambda_1 p_{12} + \lambda_3 p_{23} \\ x_3 = -\lambda_0 p_{03} - \lambda_1 p_{13} - \lambda_2 p_{23} \end{cases}$$

then we replace in F, and we get the identity $(\lambda_1 p_{01} + \lambda_2 p_{02} + \lambda_3 p_{03})^3 - (-\lambda_0 p_{01} + \lambda_2 p_{12} + \lambda_3 p_{13})(-\lambda_0 p_{02} - \lambda_1 p_{12} + \lambda_3 p_{23})(-\lambda_0 p_{03} - \lambda_1 p_{13} - \lambda_2 p_{23}) = 0$. By equating to zero the coefficients of the 20 monomials of degree 3 in $\lambda_0, \dots, \lambda_3$ we get the equations representing the lines contained in $V_P(F)$.

As a matter of fact, for this particular surface finding the lines contained in it is particularly simple. Indeed, we can distinguish the lines contained in the hyperplane "at infinity" from the lines which are projective closure of a line in $\mathbb{A}^3$. The first ones are contained in $x_0 = 0$, and it is clear that there are only three of them: $x_0 = x_1 = 0, x_0 = x_2 = 0, x_0 = x_3 = 0$. To find the others we dehomogenise F and get the equation $x_1x_2x_3 - 1 = 0$, and consider the parametrisation of a general line in $\mathbb{A}^3$: $x_i = a_it + b_i$, $i = 1, 2, 3$. By substituting, we immediately see that there are no solutions. We conclude that the surface contains only three lines.

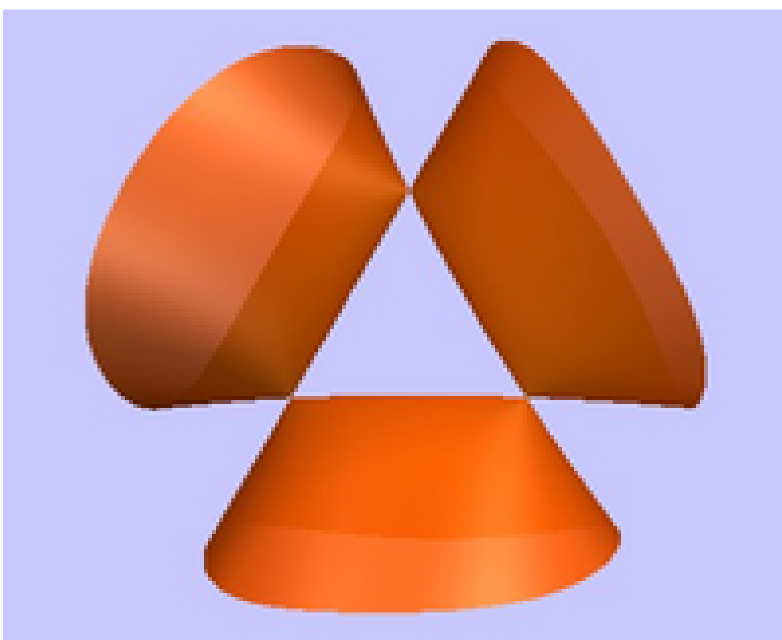

FIGURE 1. The cubic surface of Example 1.5

We consider now the restrictions to Γ_d of the two projections, and we get $\varphi_1 : \Gamma_d \to \mathbb{G}(1,n)$, $\varphi_2 : \Gamma_d \to \mathbb{P}(S_d)$. We will see now that the fibres of φ_1 are all irreducible and of the same dimension; this will allow to compute the dimension of Γ_d and get informations on the fibres of φ_2.

1. $\varphi_1(\Gamma_d) = \mathbb{G}(1,n)$, because any line ℓ is contained in some hypersurface of degree d. Indeed, up to a change of coordinates, we can assume that $\ell : x_0 = x_1 = \cdots = x_{n-2} = 0$. So $\ell \subset V_P(F)$ if and only if $F(0,\dots,0,x_{n-1},x_n) \equiv 0$, if and only if the coefficients of the monomials containing only x_{n-1}, x_n vanish, i.e. F is of the form $x_0G_0 + \cdots + x_{n-2}G_{n-2}$. So $\varphi_1^{-1}(\ell)$ is a linear subspace of dimension $N-(d+1)$, because the $d+1$ monomials $x_{n-1}^d, x_{n-1}^{d-1}x_n, \dots, x_n^d$ don't appear in F. In particular we have that the fibres of φ_1 are all irreducible and of the same dimension. By applying Proposition 1.3, we obtain that Γ_d is irreducible of dimension $\dim \mathbb{G}(1,n) + \dim \varphi_1^{-1}(\ell) = 2(n-1) + N - (d+1)$.

2. Consider now $\varphi_2 : \Gamma_d \to \mathbb{P}(S_d) = \mathbb{P}^N$. If $\dim \Gamma_d < N$, then φ_2 cannot be surjective. This happens if

$$\dim(\Gamma_d) = 2(n-1) + N - (d+1) < N \text{ if and only if } d > 2n-3.$$

We have proved the following theorem.

Theorem 1.6. *If $d > 2n-3$, there is an open non-empty subset $U \subset \mathbb{P}(S_d)$, such that if $[F] \in U$ then the hypersurface $V_P(F)$ does not contain any line; shortly, a "general" hypersurface of degree $d > 2n-3$ in $\mathbb{P}^n$ does not contain any line. The hypersurfaces containing a line form a proper closed subset in $\mathbb{P}(S_d)$.*

Example 1.7. *Let $n = 3$, the case of surfaces in $\mathbb{P}^3$. Theorem 1.6 says that a general surface of degree ≥ 4 does not contain lines. Let us analyse the cases $d = 1, 2, 3$.*

- *$d = 1$: the surface is a plane, the lines contained in a plane form a $\mathbb{P}^2$.*
- *$d = 2$: the surface is a quadric, any quadric contains lines, and precisely, if its rank is 4, it contains two families of dimension 1 parametrised by two conics in $\mathbb{G}(1,3)$; if the rank is 3, the quadric is a cone, and it contains a family of dimension 1 of lines, parametrised by a conic in $\mathbb{G}(1,3)$. In both cases of rank $3,4$ the fibres of φ_2 have dimension 1. If the rank is 2 or 1, the quadric is a pair of distinct planes or one plane with multiplicity 2, and the fibres of φ_2 have dimension 2.*
- *$d = 3$: in this case $N = 19 = \dim \Gamma_d$. Two cases can occur: either φ_2 is surjective, and a general fibre has dimension 0, or it is not surjective. In the second case, $\varphi_2(\Gamma_3)$, the variety of the cubic surfaces containing at least one line, has dimension < 19, so the fibres of $\Gamma_3 \to \varphi_2(\Gamma_3)$ have all dimension > 0. Hence, if a cubic surface contains a line, it contains by consequence infinitely many lines. But in Example 1.5 we have seen an explicit example of a cubic surface containing finitely many lines; this shows that the first possibility occurs, i.e.*

a "general" cubic surface contains finitely many lines. Theorem 1.1 explains the meaning of the adjective "general": it means that the property holds true in an open dense subset of $\mathbb{P}^{19}$.

It is a classical fact that any smooth cubic surface contains exactly 27 lines, whose configuration is completely described (see for instance [Hartshorne]). Figure 2 shows the Clebsch cubic surface, the only one having 27 real lines. In particular, among these 27 lines there are many pairs of skew lines.

It is a nice application of the theory we have developed so far to prove that such a cubic surface is rational.

Theorem 1.8. *Let* $S \subset \mathbb{P}^3$ *be a cubic surface containing two skew lines. Then* S *is rational.*

Proof. Let ℓ, ℓ' be two skew lines contained in S. For any point $P \in \mathbb{P}^3$, $P \notin \ell \cup \ell'$, there is exactly one line r_P passing through P and meeting both ℓ and ℓ': r_P is the intersection of the two planes passing through P and containing ℓ and ℓ' respectively. So we can consider the rational map $f : \mathbb{P}^3 \dashrightarrow \ell \times \ell' \simeq \mathbb{P}^1 \times \mathbb{P}^1$, such that $f(P) = (r_P \cap \ell, r_p \cap \ell')$, the pair of points of intersection of r_P with ℓ and ℓ'. We consider now the restriction $\bar{f}$ of f to S, and we get a birational map. Indeed, for any pair of points $x \in \ell$ and $x' \in \ell'$, the line joining x and x', if not contained in S, meets S in a third point. Since not all lines meeting ℓ and ℓ' can be contained in S, this defines the rational inverse of $\bar{f}$. Therefore S is birational to $\mathbb{P}^1 \times \mathbb{P}^1$, that is birational to $\mathbb{P}^2$. By transitivity we conclude that S is rational. □

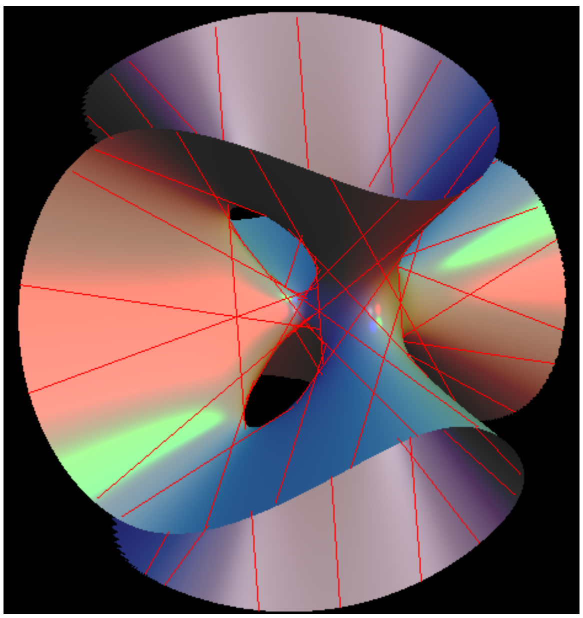

FIGURE 2. The Clebsch cubic surface

Possible equations for the Clebsch cubic surface, for different choices of coordinates, are

$$x^2y + y^2z + z^2w + w^2x = 0$$

or

$$x_0 + x_1 + x_2 + x_3 + x_4 = x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0.$$

The following equation represents the Cayley cubic surface with 4 singular points of multiplicity 2, containing 9 lines

$$xyz + yzw + zwx + wxy = 0.$$

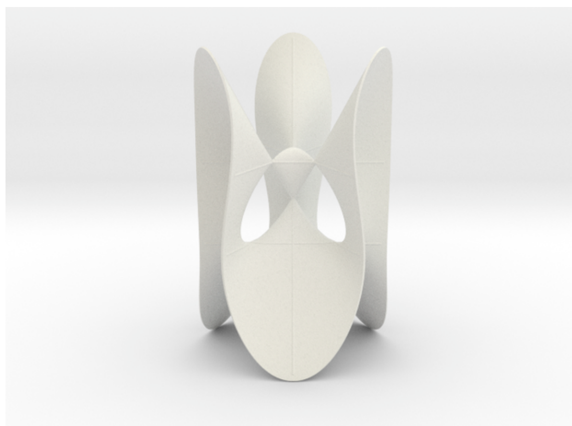

FIGURE 3. The Cayley cubic surface

A list of all possible types of singularities of cubic surfaces, with figures, can be found in the following web page: https://singsurf.org/parade/Cubics.php