– Transmission of classical information through a quantum channel

- Alice wants to transmit messages $w \in \mathcal{W} = \{1, 2, \dots, 2^{nR}\}$ to Bob through the action of a memoryless quantum channel Φ.
- She encodes $w \in \mathcal{W}$ into an n-qubit state $\rho_w^{(n)}$ that is turned into $\sigma_w^{(n)} := \Phi^{(n)}[\rho_w^{(n)}] = \underbrace{\Phi \otimes \Phi \otimes \dots \otimes \Phi}_{n \text{ times}} [\rho_w^{(n)}]$ by n-uses of the channel.
- Bob tries to retrieve w by performing a POVM measurement $\{E_w\}_{w \in \mathcal{W}}$

$$\boxed{\sum_{w \in \mathcal{W}} E_w^\dagger E_w = 1} \;.$$

- The probability of inferring w correctly is $\boxed{\mathrm{Tr}\left(\sigma_w^{(n)} E_w\right)}$
- The average probability of error is then

$$\boxed{p_{av} = \frac{1}{2^{nR}} \sum_{w \in \mathcal{W}} \left(1 - \mathrm{Tr}\left(\sigma_w^{(n)} E_w\right)\right)}$$

Definition 2.4.4.

The transmission of classical information at rate R through the quantum channel Φ is <u>reliable</u> if $p_{\omega}^{(n)} \xrightarrow[n]{} 0$.

The rate R is <u>achievable</u> if there exist a $(2^{nR}, n)$ quantum encoding and POVM's $\{E_W^{(n)}\}_{W \in \mathcal{W}}$ such that R is reliable.

The capacity of Φ is the <u>maximum reliable rate</u>, the <u>maximum</u> being taken over all possible quantum encodings of $w \in \mathcal{W}$.

<u>Remark</u>: results are available only when the encoding is via product states: $w \longrightarrow \rho_W^{(n)} = \rho_{w_1} \otimes \rho_{w_2} \otimes \cdots \otimes \rho_{w_n}$; the capacity of the quantum channel is then called <u>product state capacity</u> and denoted by $C^{(1)}(\Phi)$.

Theorem 2.4.2 <u>Holevo - Schumacher - Westmoreland</u>

The product state capacity of a quantum memoryless channel Φ is given by $C^{(1)}(\Phi) = \chi(\Phi)$, where

$$\chi(\Phi) := \max_{\{p_x, \rho_x\}} \chi(\{p_x, \Phi[\rho_x]\}) ,$$

$$\chi(\{p_x, \Phi[\rho_x]\}) := S\Big(\Phi\Big[\sum_x p_x \rho_x\Big]\Big) - \sum_x p_x S(\Phi[\rho_x]) ,$$

the maximum being evaluated over all possible statistical ensembles $\{p_x, \rho_x\}_{x \in \mathcal{X}}$, $p_x \geq 0$; $\sum_{x \in \mathcal{X}} p_x = 1$.

<u>Remark</u>

If $\rho(x) \in M_d(\mathbb{C})$ $\forall\, x \in \mathcal{X}$ then one can vary over ensembles consisting of at most d^2 pure states. (Try to prove it)

Proof that $R > \chi(\Phi)$ implies $p_{av}^{(n)} \underset{n\to+\infty}{\not\to} 0$.

- Alice's Encoding: $\{1, 2, \dots, 2^{nR}\} \ni w \longmapsto \boxed{\rho_w^{(n)} = \rho_1(w) \otimes \dots \otimes \rho_n(w)}$

$$\rho_i(w) \in M_d(\mathbb{C}) \,, \quad \rho_w^{(n)} \in M_d(\mathbb{C})^{\otimes n} = \underbrace{M_d(\mathbb{C}) \otimes \dots \otimes M_d(\mathbb{C})}_{n \text{ times}}$$

- Channel action: $\rho_w^{(n)} \longrightarrow \boxed{\sigma_w^{(n)} = \Phi^{\otimes n}\left[\rho_w^{(n)}\right] = \sigma_1(w) \otimes \dots \otimes \sigma_n(w)}$

$$\sigma_j(w) = \Phi[\rho_j(w)]$$

- Bob's measurement: $\boxed{\mathcal{E} := \{E_w\}_{w=0,1,\dots,2^{nR}}}$: $\boxed{\sum_{w=0}^{2^{nR}} E_w^\dagger E_w = 1}$

- Average error: $$p_{av}^{(n)} = \frac{\sum_{w=1}^{2^{nR}} \left(1 - \mathrm{Tr}\left(\sigma_w^{(n)} E_w^\dagger E_w\right)\right)}{2^{nR}}$$

- Holevo's bound : $\rho, \rho_x \in M_d(\mathbb{C}) \Rightarrow \boxed{I_{acc}(X) \le S(\rho) - \sum_{x\in\mathcal{X}} p(x) S(\rho_x) \le \log d}$

Therefore : $\log(2^{nR}) \le \log(d^n) \Rightarrow \boxed{R \le \log d}$

- Fano's inequality : $\boxed{H(W|\mathcal{E}) \le h(p_{av}^{(n)}) + p_{av}^{(n)} \log(d^n)}$

$$h(p_{av}^{(n)}) = -p_{av}^{(n)} \log p_{av}^{(n)} - (1-p_{av}^{(n)}) \log(1-p_{av}^{(n)})$$

$$H(W|\mathcal{E}) = H(W) - I(W;\mathcal{E}) = nR - I(W;\mathcal{E})$$

$$\ge nR - S\Big(\sum_{w\in\mathcal{W}} \frac{1}{2^{nR}} \sigma_w^{(n)}\Big) + \sum_{w\in\mathcal{W}} \frac{1}{2^{nR}} S(\sigma_w^{(n)}) \quad \text{(Holevo's bound)}$$

$$\ge nR - S\Big(\sum_{w\in\mathcal{W}} \frac{1}{2^{nR}} \sigma_1(w)\otimes\cdots\otimes\sigma_n(w)\Big) + \frac{1}{2^{nR}} \sum_{w\in\mathcal{W}} S(\sigma_1(w)\otimes\cdots\otimes\sigma_n(w))$$

$$\ge nR - \sum_{j=1}^{n} \Big(S\Big(\sum_{w\in\mathcal{W}} \frac{1}{2^{nR}} \sigma_j(w)\Big) - \frac{1}{2^{nR}} \sum_{w\in\mathcal{W}} S(\sigma_j(w))\Big) \quad \text{(sub-additivity)}$$

$$\ge nR - n\chi(\Phi) \quad \text{(Holevo's bound)}$$

Then, $\quad p_{av}^{(n)}\, n \log(d) \geq H(W|\mathcal{E}) - h(p_{av}^{(n)})$

$$\geq n\,(R - \chi(\Phi))$$

$$\boxed{p_{av}^{(n)} \geq \frac{R-\chi(\Phi)}{\log(d)} - \frac{h(p_{av}^{(n)})}{n\,\log(d)}} \not\longrightarrow 0 \quad \boxed{\Leftarrow\; R > \chi(\Phi)}$$

<u>Proof</u> that $R < \chi(\Phi)$ is an achievable rate.

- Idea: random coding, $\boxed{\mathrm{Prob}(\mathcal{E}) = \prod_{w\in\mathcal{W}} \prod_{j} p(x_j^{\mathcal{E}}(w))}$

$$\mathcal{W} \ni w \xrightarrow{\mathcal{E}} x_1^{\mathcal{E}}(w)\, x_2^{\mathcal{E}}(w) \cdots x_n^{\mathcal{E}}(w) =: x_{\mathcal{E}}^{(n)} \in \underbrace{\mathcal{X} \times \mathcal{X} \times \cdots \times \mathcal{X}}_{n \text{ times}}$$

$$\downarrow$$

$$\rho_{\mathcal{E}}^{(n)}(w) = \rho(x_1^{\mathcal{E}}(w)) \otimes \cdots \otimes \rho(x_n^{\mathcal{E}}(w)) \in M_d(\mathbb{C})^{\otimes n}$$

$$\downarrow$$

$$\sigma_{\mathcal{E}}^{(n)}(w) = \Phi^{\otimes n}\left[\rho_{\mathcal{E}}^{(n)}(w)\right] = \underbrace{\sigma(x_1^{\mathcal{E}}(w))}_{\Phi[\rho(x_1^{\mathcal{E}}(w))]} \otimes \cdots \otimes \sigma(x_n^{\mathcal{E}}(w))$$

- $\boxed{\mathbb{E}\left[\sigma_{\mathcal{E}}^{(n)}(w)\right]} = \sum_{\mathcal{E}} \mathrm{Prob}(\mathcal{E})\, \sigma_{\mathcal{E}}^{(n)}(w) = \bar{\sigma}^{\otimes n} = \boxed{\underbrace{\bar{\sigma} \otimes \cdots \otimes \bar{\sigma}}_{n \text{ times}}}$

$$\boxed{\bar{\sigma} = \sum_{x \in \mathcal{X}} p(x)\, \sigma(x) = \sum_{x \in \mathcal{X}} p(x)\, \Phi[\rho(x)]}$$

- $\bar{\sigma}^{\otimes n}$ typical subspace projector $P_{\varepsilon}^{(n)}$

$$\boxed{\mathrm{Tr}\left(\bar{\sigma}^{\otimes n}\left(1 - P_{\varepsilon}^{(n)}\right)\right) \leq \varepsilon}$$

- $\boxed{\sigma(x) = \sum_{j} s_j(x)\, |s_j(x)\rangle\langle s_j(x)|}$

$$\begin{cases} \sigma_{\mathcal{E}}^{(n)}(w) = \sum_{j(n)} S_{j(n)}^{\mathcal{E}}(w)\ |S_{j(n)}^{\mathcal{E}}\rangle\langle S_{j(n)}^{\mathcal{E}}| \\[2ex] S_{j(n)}^{\mathcal{E}}(w) = s_{j_1}(x_1^{\mathcal{E}}(w)) \cdots s_{j_n}(x_n^{\mathcal{E}}(w)) \\[2ex] |S_{j(n)}^{\mathcal{E}}(w)\rangle = |s_{j_1}(x_1^{\mathcal{E}}(w))\rangle \otimes \cdots \otimes |s_{j_n}(x_n^{\mathcal{E}}(w))\rangle \end{cases}$$

- $$\boxed{\bar{S} = \sum_{x \in \mathcal{X}} p(x)\, S(\sigma(x))}$$

$$\begin{cases} T^{(n)}_{\mathcal{E},\varepsilon}(w) := \mathrm{Lin\,Span}\left\{ |s^{\mathcal{E}}_{j(n)}(w)\rangle : \left| -\frac{1}{n}\log\left(s^{\mathcal{E}}_{j(n)}(w)\right) - \bar{S} \right| \le \varepsilon \right\} \\ P^{(n)}_{\mathcal{E},\varepsilon}(w) : (\mathbb{C}^d)^{\otimes n} \longrightarrow T^{(n)}_{\mathcal{E},\varepsilon}(w) \quad \underline{\text{orthogonal projection}} \end{cases}$$

$$\boxed{-\sum_{\mathcal{E}} \mathrm{Prob}(\mathcal{E}) \sum_{s^{\mathcal{E}}_{j(n)}(w)} s^{\mathcal{E}}_{j(n)}(w) \log s^{\mathcal{E}}_{j(n)}(w) = \bar{S}} \Longrightarrow \boxed{\begin{array}{l} \mathbb{E}\left[\mathrm{Tr}\left(\sigma^{(n)}_w P^{(n)}_{\mathcal{E},\varepsilon}(w)\right)\right] \ge 1-\varepsilon \\ \mathbb{E}\left[\mathrm{Tr}\left(P^{(n)}_{\mathcal{E},\varepsilon}(w)\right)\right] \le 2^{n(\bar{S}+\varepsilon)} \end{array}}$$

- Bob's <u>square root decoding</u>

$$\boxed{E_w := P^{(n)}_{\mathcal{E},\varepsilon}(w)\, P^{(n)}_{\varepsilon} \frac{1}{\sqrt{\sum_{\bar{w}} P^{(n)}_{\varepsilon} P^{(n)}_{\mathcal{E},\varepsilon}(\bar{w})\, P^{(n)}_{\varepsilon}}}} \quad , \quad \sum_{\bar{w}} P^{(n)}_{\varepsilon} P^{(n)}_{\mathcal{E},\varepsilon}(\bar{w}) P^{(n)}_{\varepsilon} =: \Pi^{(n)}_{\mathcal{E},\varepsilon}$$

$$\sum_w E^\dagger_w E_w = \frac{1}{\sqrt{\Pi^{(n)}_{\mathcal{E},\varepsilon}}}\left(\sum_w P^{(n)}_{\varepsilon} P^{(n)}_{\mathcal{E},\varepsilon}(w)\, P^{(n)}_{\varepsilon}\right)\frac{1}{\sqrt{\Pi^{(n)}_{\mathcal{E},\varepsilon}}} = \mathbb{1}$$

- Short-hand notation: $\begin{cases} s_j^{\cdot}(w) := s^{\varepsilon}_{j(n)}(w) \ , \ |s^{\varepsilon}_{j(n)}(w)\rangle =: |s_j^{\cdot}(w)\rangle \\ P_w := P^{(n)}_{\varepsilon,\varepsilon}(w) = \sum_j{}' |s_j^{\cdot}(w)\rangle\langle s_j^{\cdot}(w)| \ , \ P := P^{(n)}_{\varepsilon} \\ \Pi = \sum_w P\, P_w\, P \ , \ |\tilde{s}_j^{\cdot}(w)\rangle := P|s_j^{\cdot}(w)\rangle \end{cases}$

$$\operatorname{Tr}\left(\sigma^{(n)}_w E_w^{+} E_w\right) = \sum_k s_k(w) \, \langle \tilde{s}_k(w)| \frac{1}{\sqrt{\Pi}} \sum_e{}' |\tilde{s}_e(w)\rangle\langle\tilde{s}_e(w)| \frac{1}{\sqrt{\Pi}} |\tilde{s}_k(w)\rangle$$

$$= \sum_k \sum_e{}' s_k(w) \left| \langle \tilde{s}_k(w)| \frac{1}{\sqrt{\Pi}} |\tilde{s}_e(w)\rangle \right|^2$$

$$\geq \sum_k{}' \sum_e{}' s_k(w) \left| \langle \tilde{s}_k(w)| \frac{1}{\sqrt{\Pi}} |\tilde{s}_e(w)\rangle \right|^2$$

$$p^{(n)}_{ev} = \frac{1}{2^{nR}} \sum_w \left(1 - \operatorname{Tr}\left(\sigma^{(n)}_w E_w^{+} E_w\right)\right) \leq \frac{1}{2^{nR}} \sum_w \left(\sum_k{}' s_k(w) \left(1 - \sum_e{}' \left|\langle \tilde{s}_k(w)| \frac{1}{\sqrt{\Pi}} |\tilde{s}_e(w)\rangle\right|^2\right) + \sum_k{}'' s_k(w)\right)$$

$$= \frac{1}{2^{nR}} \sum_w \left(\sum_k{}' s_k(w) \left(1 - \sum_e{}' \left|\langle \tilde{s}_k(w)| \frac{1}{\sqrt{\Pi}} |\tilde{s}_e(w)\rangle\right|^2 + \operatorname{Tr}\left(\sigma^{(n)}_w \left(1 - P^{(n)}_{\varepsilon,\varepsilon}(w)\right)\right)\right)\right)$$

- $$P_{av}^{(n)} \leq \frac{1}{2^{nR}} \sum_{w} \left({\sum_{k}}' s_k(w) \left(1 - \left|\langle \tilde{s}_k(w)| \frac{1}{\sqrt{\Pi}} |\tilde{s}_k(w)\rangle\right|^2\right) + \mathrm{Tr}\left(\sigma_w^{(n)}\left(1 - P_{\varepsilon,\varepsilon}^{(n)}(w)\right)\right)\right)$$

- $$M_{kw;k'w'} := \langle \tilde{s}_k(w)| \frac{1}{\sqrt{\Pi}} |\tilde{s}_{k'}(w')\rangle \quad ; \quad \Pi = \sum_{\bar{w}} \sum_{k} |\tilde{s}_k(\bar{w})\rangle\langle\tilde{s}_k(\bar{w})|$$

$$\boxed{\sum_{k',w'} M_{kw;k'w'} \, M_{k'w';k''w''} = \langle \tilde{s}_k(w)|\tilde{s}_{k''}(w'')\rangle}$$

$$\langle \tilde{s}_k(w)|\tilde{s}_k(w)\rangle = \langle s_k(w)|P|s_k(w)\rangle \leq 1$$

$$\|$$

$$\sum_{k',w'} M_{kw;k'w'} \, M_{k'w';kw} \geq \left(M_{kw;kw}\right)^2 = \left|\langle \tilde{s}_k(w)| \frac{1}{\sqrt{\Pi}} |\tilde{s}_k(w)\rangle\right|^2$$

$$\boxed{(1-x)^2 = (1-x)(1+x) \leq 2(1-x) \Longleftarrow 0 \leq x \leq 1}$$

- $$P_{av}^{(n)} \leq \frac{1}{2^{nR}} \sum_{w} \left(2 {\sum_{k}}' s_k(w) \left(1 - \langle \tilde{s}_k(w)| \frac{1}{\sqrt{\Pi}} |\tilde{s}_k(w)\rangle\right) + \mathrm{Tr}\left(\sigma_w^{(n)}\left(1 - P_{\varepsilon,\varepsilon}^{(n)}(w)\right)\right)\right)$$

- $M := [M_{kw,k'w'}]$: $2(1-M) = (1-M)^2 + 1 - M^2 = (1-M^2)^2 \frac{1}{1+M^2} + 1 - M^2$

$$\le (1-M^2)^2 + 1 - M^2 = 2 - 3M^2 + M^4 \le 2(1-M^2) + M^4$$

- $$P_{ew}^{(n)} \le \frac{1}{2^{nR}} \sum_w \Big(2 \sum_k{}' s_k(w) \big(1 - \langle \tilde{s}_k(w) | \tilde{s}_k(w) \rangle\big) + \sum_k{}' s_k(w) \sum_{k'}{}' \sum_{w'} |\langle \tilde{s}_k(w) | \tilde{s}_{k'}(w') \rangle|^2$$
$$+ \mathrm{Tr}\big(\sigma_w^{(n)} (1 - P_{\xi,\varepsilon}^{(n)}(w)) \big) \Big)$$

$$\le \frac{1}{2^{nR}} \sum_w \Big(2\, \mathrm{Tr}\big(\sigma_w^{(n)} (1 - P_\varepsilon^{(n)}) \big) + \mathrm{Tr}\big(\sigma_w^{(n)} (1 - P_{\xi,\varepsilon}^{(n)}(w)) \big)$$
$$+ \sum_{k,k'}{}' s_k(w) |\langle \tilde{s}_k(w) | \tilde{s}_{k'}(w) \rangle|^2$$
$$+ \sum_{k,k'}{}' s_k(w) \sum_{w' \neq w} |\langle \tilde{s}_k(w) | \tilde{s}_k(w') \rangle|^2 \Big)$$

- $\sum_{k'}' |\langle \tilde{s}_k(w) | \tilde{s}_{k'}(w) \rangle|^2 = \sum_{k'}' \langle s_k(w) | P | s_{k'}(w) \rangle \langle s_{k'}(w) | P | s_k(w) \rangle$

$$= \sum_{k'}' \langle s_k(w) | (1-P) | s_{k'}(w) \rangle \langle s_{k'}(w) | (1-P) | s_k(w) \rangle$$

$$= \langle s_k(w) | (1-P) P_w (1-P) | s_k(w) \rangle$$

- $\sum_{k'}' \sum_{w' \neq w} |\langle \tilde{s}_k(w) | \tilde{s}_{k'}(w') \rangle|^2 = \sum_{w' \neq w} \langle s_k(w) | P P_{w'} P | s_k(w) \rangle$

- $P_{ov}^{(n)} \leq \frac{1}{2^{nR}} \sum_w \Big(2\,\mathrm{Tr}\big(\sigma_w^{(n)} (1 - P_\varepsilon^{(n)})\big) + \mathrm{Tr}\big(\sigma_w^{(n)} (1 - P_{\epsilon,\varepsilon}^{(n)}(w))\big)$

$$+ \mathrm{Tr}\Big(\sigma_w^{(n)} (1 - P_\varepsilon^{(n)}) P_{\epsilon,\varepsilon}^{(n)}(w) (1 - P_\varepsilon^{(n)})\Big) \qquad \Big(\leq \mathrm{Tr}\big(\sigma_w^{(n)} (1 - P_\varepsilon^{(n)})\big)\Big)$$

$$+ \sum_{\bar{w} \neq w} \mathrm{Tr}\Big(\sigma_w^{(n)} \big(P_\varepsilon^{(n)} P_{\epsilon,\varepsilon}^{(n)}(\bar{w}) P_\varepsilon^{(n)}\big)\Big)\Big)$$

- $$\boxed{p_{av}^{(n)} \leq \frac{1}{2^{nR}} \sum_{w} \left(3\,\mathrm{Tr}\left(\sigma_w^{(n)}(1 - P_\varepsilon^{(n)})\right) + \mathrm{Tr}\left(\sigma_w^{(n)}(1 - P_{\zeta,\varepsilon}^{(n)}(w))\right) + \sum_{\bar{w}\neq w} \mathrm{Tr}\left(\sigma_w^{(n)}\left(P_\varepsilon^{(n)} P_{\zeta,\varepsilon}^{(n)}(\bar{w}) P_\varepsilon^{(n)}\right)\right)\right)}$$

- Expectation with respect to random cooling

$$\mathbb{E}[p_{av}^{(n)}] \leq 3\,\mathrm{Tr}\left(\bar{\sigma}^{\otimes n}(1 - P_\varepsilon^{(n)})\right) + \mathbb{E}\left(\mathrm{Tr}\left(\sigma_1^{(n)}(1 - P_{\zeta,\varepsilon}^{(n)}(1))\right)\right) + (2^{nR}-1)\,\mathrm{Tr}\left(\bar{\sigma}^{\otimes n} P_\varepsilon^{(n)}\, \mathbb{E}[P_{\zeta,\varepsilon}^{(n)}(1)]\, P_\varepsilon^{(n)}\right)$$

$$\leq 4\varepsilon + (2^{nR}-1)\,\mathrm{Tr}\left(\bar{\sigma}^{\otimes n} P_\varepsilon^{(n)}\, \mathbb{E}[P_{\zeta,\varepsilon}^{(n)}(1)]\, P_\varepsilon^{(n)}\right)$$

- $$\boxed{P_\varepsilon^{(n)}\, \bar{\sigma}^{\otimes n}\, P_\varepsilon^{(n)} \leq 2^{-n(S(\bar{\sigma})-\varepsilon)}} \; ; \; \boxed{\mathrm{Tr}\, \mathbb{E}[P_{\zeta,\varepsilon}^{(n)}(1)] = \mathbb{E}\left[\mathrm{Tr}\left(P_{\zeta,\varepsilon}^{(n)}(1)\right)\right] \leq 2^{n(\bar{S}+\varepsilon)}}$$

- $$\mathbb{E}[p_{av}^{(n)}] \leq 4\varepsilon + (2^{nR}-1)\, 2^{-n(S(\bar{\sigma})-\varepsilon)}\, 2^{n(\bar{S}+\varepsilon)}$$

$$\leq 4\varepsilon + 2^{-n(S(\bar{\sigma}) - \bar{S} - R)}$$

$$\boxed{\begin{aligned} \bar{\sigma} &= \sum_{x\in\mathcal{X}} p(x)\, \Phi[\rho(x)] \\ \bar{S} &= \sum_{x\in\mathcal{X}} p(x)\, S(\Phi[\rho(x)]) \end{aligned}}$$