LESSON 19.

1. Grassmannians.

In this Lesson we will see how the antisymmetric tensors play an important role in algebraic geometry, providing an ambient space in which naturally embeds the Grassmannian of subspaces of fixed dimension of a vector space, or, equivalently, of a projective space.

To define the exterior powers of the vector space V, one proceeds in a way which is similar to the one used to define its symmetric powers. We define the d-th exterior power $\wedge^d V$ as the quotient $V^{\otimes d}/\Lambda$, where Λ is generated by the tensors of the form $v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_d$, with $v_i = v_j$ for some $i \neq j$. The following notation is used: $[v_1 \otimes \cdots \otimes v_d] = v_1 \wedge \cdots \wedge v_d$.

There is a natural multilinear alternating map $V \times \cdots \times V = V^d \to \wedge^d V$, that enjoys the universal property. Given a basis $\mathcal{B} = (e_1, \dots, e_n)$ of V, a basis of $\wedge^d V$ is formed by the tensors $e_{i_1} \wedge \dots \wedge e_{i_d}$, with $1 \leq i_1 < \dots < i_d \leq n$. Therefore $\dim \wedge^d V = \binom{n}{d}$. The exterior algebra of V is the following direct sum: $\wedge V = \oplus_{d \geq 0} \wedge^d V = K \oplus V \oplus \wedge^2 V \oplus \dots$. To define an inner product that gives it the structure of algebra we can proceed as follows.

Step 1. Fixed $v_1, \dots, v_q \in V$, define $f : V^d \to \wedge^{d+p} V$ posing $f(x_1, \dots, x_d) = x_1 \wedge \dots \wedge x_d \wedge v_1 \wedge \dots \wedge v_q$. Since f results to be multilinear and alternating, by the universal property we get a factorization of f through $\wedge^d V$, which gives a linear map $\bar{f} : \wedge^d V \to \wedge^{d+p} V$, extending f. For any $\omega \in \wedge^d V$, we denote $\bar{f}(\omega)$ by $\omega \wedge v_1 \wedge \dots \wedge v_d$.

Step 2. Fixed $\omega \in \wedge^d V$, consider the map $g : V^p \to \wedge^{d+p} V$ such that $g(y_1, \dots, y_p) = \omega \wedge y_1 \wedge \dots \wedge y_p$: it is multilinear and alternating, therefore it factorizes through $\wedge^p V$ and we get a linear map $\bar{g} : \wedge^p V \to \wedge^{d+p} V$, extending g. We denote $\bar{g}(\sigma) := \omega \wedge \sigma$.

Step 3. For any $d, p \geq 0$ we have got a map $\wedge : \wedge^d V \times \wedge^p V \to \wedge^{d+p} V$, that results to be bilinear, and extends to an inner product $\wedge : (\wedge V) \times (\wedge V) \to \wedge V$, which gives $\wedge V$ the required structure of algebra. It is a graded algebra, the non-zero homogeneous components are those of degree from 0 to $n = \dim V$.

Proposition 1.1. *Let V be a vector space of dimension n.*

(i) Vectors $v_1, \dots, v_p \in V$ are linearly dependent if and only if $v_1 \wedge \dots \wedge v_p = 0$.

(ii) Let $v \in V$ be a non-zero vector, and $\omega \in \wedge^p V$. Then $\omega \wedge v = 0$ if and only if there exists $\Phi \in \wedge^{p-1} V$ such that $\omega = \Phi \wedge v$. In this case we say that v divides ω.

Proof. The proof of (i) is standard. If $\omega = \Phi \wedge v$, then $\omega \wedge v = (\Phi \wedge v) \wedge v = \Phi \wedge (v \wedge v) = 0$. Conversely, if $\omega \wedge v = 0$, $v \neq 0$, we choose a basis of V, $\mathcal{B} = (e_1, \dots, e_n)$ with $e_1 = v$. Write $\omega = \Sigma_{i_1 < \cdots < i_p} a_{i_1 \dots i_p} e_{i_1} \wedge \dots \wedge e_{i_p}$. Then $0 = \omega \wedge e_1 = \Sigma_{i_1 < \cdots < i_p} (\pm) a_{i_1 \dots i_p} e_1 \wedge e_{i_1} \wedge \dots \wedge e_{i_p}$.

If $i_1 = 1$, the corresponding summand does not appear in this sum, so it remains a linear combination of linearly independent tensors, which implies $a_{i_1 \dots i_p} = 0$ every time $i_1 > 1$. Therefore $\omega = e_1 \wedge \Phi$ for a suitable Φ. $\square$

Proposition 1.2. *Let $\omega \neq 0$ be an element of $\wedge^p V$. Then ω is totally decomposable if and only if the subspace of V: $W = \{v \in V \mid v \text{ divides } \omega\}$ has dimension p.*

Proof. If $\omega = x_1 \wedge \dots \wedge x_p \neq 0$, then $x_1, \dots, x_p$ are linearly independent and belong to W. So we can extend them to a basis of V adding vectors $x_{p+1}, \dots, x_n$. If $v \in W$, $v = \alpha_1 x_1 + \dots + \alpha_n x_n$, and v divides ω, then $\omega \wedge v = 0$, i.e. $x_1 \wedge \dots \wedge x_p \wedge (\alpha_1 x_1 + \dots + \alpha_n x_n) = 0$. This implies $\alpha_{p+1} x_1 \wedge \dots \wedge x_p \wedge x_{p+1} + \dots + \alpha_n x_1 \wedge \dots \wedge x_p \wedge x_n = 0$, therefore $\alpha_{p+1} = \dots = \alpha_n = 0$, so $v \in \langle x_1, \dots, x_p \rangle$.

Conversely, if $(x_1, \dots, x_p)$ is a basis of W, we can complete it to a basis of V and write $\omega = \Sigma a_{i_1 \dots i_p} x_{i_1} \wedge \dots \wedge x_{i_p}$. But x_1 divides ω, so $\omega \wedge x_1 = 0$. Replacing ω with its explicit expression, we obtain that $a_{i_1 \dots i_p} = 0$ if $1 \notin \{i_1, \dots, i_p\}$. Repeating this argument for $x_2, \dots, x_p$, it remains $\omega = a_{1 \dots p} x_1 \wedge \dots \wedge x_p$. $\square$

With explicit computations, one can prove the following proposition.

Proposition 1.3. *Let V be a vector space with $\dim V = n$. Let $\mathcal{B} = (e_1, \dots, e_n)$ be a basis of V and $v_1, \dots, v_n$ be any vectors. Then $v_1 \wedge \dots \wedge v_n = \det(A) e_1 \wedge \dots \wedge e_n$, where A is the matrix of the coordinates of the vectors $v_1, \dots, v_n$ with respect to $\mathcal{B}$.*

Corollary 1.4. *Let $v_1, \dots, v_p \in V$, with $v_i = \Sigma a_{ij} e_j$, $i = 1, \dots, p$. Then $v_1 \wedge \dots \wedge v_p = \Sigma_{i_1 < \dots < i_p} a_{i_1 \dots i_p} e_{i_1} \wedge \dots \wedge e_{i_p}$, with $a_{i_1 \dots i_p} = \det(A_{i_1 \dots i_p})$, the determinant of the $p \times p$ submatrix of A containing the columns of indices $i_1, \dots, i_p$.*

We are now ready to introduce the Grassmannian and to give it an interpretation as projective variety via the Plücker map. Let V be a vector space of dimension n, and r be a positive integer, $1 \leq r \leq n$. The Grassmannian $G(r, V)$ is the set whose elements are the subspaces of V of dimension r. It is usual also to denote it by $G(r, n)$.

There is a natural bijection between $G(r, V)$ and the set of the projective subspaces of $\mathbb{P}(V)$ of dimension $r - 1$, denoted by $\mathbb{G}(r-1, \mathbb{P}(V))$ or $\mathbb{G}(r-1, n-1)$. Let $W \in G(r, V)$; if $(w_1, \dots, w_r)$ and $(x_1, \dots, x_r)$ are two bases of W, then $w_1 \wedge \dots \wedge w_r = \lambda x_1 \wedge \dots \wedge x_r$, where $\lambda \in K$ is the determinant of the matrix of the change of basis. Therefore W uniquely determines an element of $\wedge^r V$ up to proportionality. This allows to define a map, called the Plücker map, $\psi : G(r, V) \to \mathbb{P}(\wedge^r V)$, such that $\psi(W) = [w_1 \wedge \dots \wedge w_r]$.

Proposition 1.5. *The Plücker map is injective.*

Proof. Assume $\psi(W) = \psi(W')$, where W, W' are subspaces of V of dimension r with bases $(x_1, \dots, x_r)$ and $(y_1, \dots, y_r)$. So there exists $\lambda \neq 0$ in K such that $x_1 \wedge \cdots \wedge x_r = \lambda y_1 \wedge \cdots \wedge y_r$. This implies $x_1 \wedge \cdots \wedge x_r \wedge y_i = 0$ for any i, so y_i is linearly dependent from $x_1, \dots, x_r$, so $y_i \in W$. Therefore $W' \subset W$. The reverse inclusion is similar. □

In coordinates with respect to the basis of $\wedge^r V$ $\{e_{i_1} \wedge \ldots \wedge e_{i_r}, 1 \leq i_1 < \ldots < i_r \leq n\}$, $\psi(W)$ is given by the minors of maximal order r of the matrix of the coordinates of the vectors of a basis of W, with respect to $e_1, \dots, e_n$.

Example 1.6.

(i) $r = n-1$: $\wedge^{n-1} V$ has dimension n. It results to be isomorphic to the dual vector space V^*, and an explicit isomorphism is obtained associating to $e_1 \wedge \cdots \wedge \hat{e}_k \wedge \cdots \wedge e_n$ the linear form e_k^* of the dual basis. In this case the Plücker map is surjective, so $\psi(G(n-1, n)) \simeq \mathbb{P}(V^*)$.

(ii) $n = 4, r = 2$: $G(2,4)$ or $\mathbb{G}(1,3)$, the Grassmannian of lines in $\mathbb{P}^3$. In this case $\psi : \mathbb{G}(1,3) \to \mathbb{P}(\wedge^2 V) \simeq \mathbb{P}^5$. Let (e_0, e_1, e_2, e_3) be a basis of V. Let $\ell = \mathbb{P}(L)$ be the line of $\mathbb{P}^3$ obtained by projectivisation of the vector subspace $L \subset V$ of dimension 2, let $L = \langle x, y \rangle$; then $\psi(\ell) = [x \wedge y]$. Its Plücker coordinates are traditionally denoted by $p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23}$, with $p_{ij} = x_i y_j - x_j y_i$, the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{pmatrix}.$$

This time ψ is not surjective; its image is the subset of $\wedge^2 V$ of the totally decomposable tensors. Assume $char(K) \neq 2$. They satisfy the equation of degree 2: $p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0$, which represents a quadric of maximal rank in $\mathbb{P}^5$, called the Klein quadric. The fact that this equation is satisfied can be seen by considering the 4×4 matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{pmatrix}:$$

its determinant is precisely the above equation (consider the development of the determinant according to the first two rows).

For instance the line of equations $x_2 = x_3 = 0$, obtained projectivising the subspace $\langle e_0, e_1 \rangle$, has Plücker coordinates $[1, 0, 0, 0, 0, 0]$.

In general we can prove the following theorem.

Theorem 1.7. *The image of the Plücker map is a closed subset in* $\mathbb{P}(\wedge^r V)$.

Proof. The image of the Plücker map is the set of the proportionality classes of totally decomposable tensors. By Proposition 1.2, a tensor $\omega \in \wedge^r V$ is totally decomposable if and only if the subspace $W = \{v \in V \mid v \text{ divides } \omega\}$ has dimension r. We consider the linear map $\Phi : V \to \wedge^{r+1} V$, such that $\Phi(v) = \omega \wedge v$. The kernel of Φ is equal to W. So ω is totally decomposable if and only if the rank of Φ is $n-r$. Fixed a basis $\mathcal{B} = (e_1, \ldots, e_n)$ of V, we write $\omega = \Sigma_{i_1 < \cdots < i_r} a_{i_1 \ldots i_r} e_{i_1} \wedge \ldots \wedge e_{i_r}$. We then consider the basis of $\wedge^{r+1} V$ associated to $\mathcal{B}$ and we construct the matrix A of Φ with respect to these bases: its minors of order $n-r+1$ are equations of the image of ψ, and they are polynomials in the coordinates $a_{i_1 \ldots i_r}$ of ω. □

From now on we shall identify the Grassmannian with the projective algebraic set that is its image in the Plücker map. The equations obtained in Theorem 1.7 are nevertheless not generators for the ideal of the Grassmannian. For instance, in the case $n = 4, r = 2$, let $\omega = p_{01} e_0 \wedge e_1 + p_{02} e_0 \wedge e_2 + \ldots$. Then:

$\Phi(e_0) = \omega \wedge e_0 = p_{12} e_0 \wedge e_1 \wedge e_2 + p_{13} e_0 \wedge e_1 \wedge e_3 + p_{23} e_0 \wedge e_2 \wedge e_3;$
$\Phi(e_1) = \omega \wedge e_1 = -p_{02} e_0 \wedge e_1 \wedge e_2 - p_{03} e_0 \wedge e_1 \wedge e_3 + p_{23} e_1 \wedge e_2 \wedge e_3;$
$\Phi(e_2) = \omega \wedge e_2 = p_{01} e_0 \wedge e_1 \wedge e_2 - p_{03} e_0 \wedge e_2 \wedge e_3 + p_{13} e_1 \wedge e_2 \wedge e_3;$
$\Phi(e_3) = \omega \wedge e_3 = p_{01} e_0 \wedge e_1 \wedge e_3 + p_{02} e_0 \wedge e_2 \wedge e_3 + p_{12} e_1 \wedge e_2 \wedge e_3.$

So the matrix is

$$\begin{pmatrix} p_{12} & -p_{02} & p_{01} & 0 \\ p_{13} & -p_{03} & 0 & p_{01} \\ p_{23} & 0 & -p_{03} & p_{02} \\ 0 & p_{23} & p_{13} & p_{12} \end{pmatrix}.$$

Its 3×3 minors are equations defining $\mathbb{G}(1,3)$, but the radical of the ideal generated by these minors is in fact $(p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12})$.

To find equations for the Grassmannian and to prove that it is irreducible, it is convenient to give an explicit open covering with affine open subsets. In $\mathbb{P}(\wedge^r V)$, let $U_{i_1 \ldots i_r}$ be the affine open subset where the Plücker coordinate $p_{i_1 \ldots i_r} \neq 0$. To simplify notation we assume $i_1 = 1, i_2 = 2, \ldots, i_r = r$, and we put $U = U_{1 \ldots r}$. If $W \in G(r,n) \cap U$, and $w_1, \ldots, w_r$ is a basis of W, then the first minor of the matrix M of the coordinates of $w_1, \ldots, w_r$ is non-degenerate. So we can choose a new basis of W such that M is of the form

$$M = \begin{pmatrix} 1 & 0 & \ldots & 0 & \alpha_{1,r+1} & \ldots & \alpha_{1,n} \\ 0 & 1 & \ldots & 0 & \alpha_{2,r+1} & \ldots & \alpha_{2,n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & \alpha_{r,r+1} & \ldots & \alpha_{r,n} \end{pmatrix}.$$

Conversely, any matrix of this form defines a subspace $W \in G(r,n) \cap U$. So there is a bijection between $G(r,n) \cap U$ and $K^{r(n-r)}$, i.e. the affine space of dimension $r(n-r)$. The coordinates of W result to be equal to 1 and all minors of all orders of the submatrix of the last $n-r$ columns of M. Therefore they are expressed as polynomials in the $r(n-r)$ elements of the last $n-r$ columns of M. This shows that $G(r,n)\cap U$ is an affine subvariety of U isomorphic to $\mathbb{A}^{r(n-r)}$. By homogenising the equations obtained in this way, one gets equations for $G(r,n)$.

For instance, in the case $n=4, r=2$, the matrix M becomes

$$M = \begin{pmatrix} 1 & 0 & \alpha_{13} & \alpha_{14} \\ 0 & 1 & \alpha_{23} & \alpha_{24} \end{pmatrix}.$$

One gets $1 = p_{01}, \alpha_{23} = p_{02}, \alpha_{24} = p_{03}, -\alpha_{13} = p_{12}, -\alpha_{14} = p_{13}, \alpha_{13}\alpha_{24} - \alpha_{23}\alpha_{14} = p_{23}$. If we make the substitutions and homogenise the last equation with respect to p_{01}, we find the equation of the Klein quadric $p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0$.

Theorem 1.8. *$G(r,n)$ is an irreducible projective variety of dimension $r(n-r)$, and it is rational.*

Proof. We remark that $G(r,n)\cap U_{i_1\dots i_r}$ is the set of the subspaces W which are complementar to the subspace of equations $x_{i_1} = \ldots = x_{i_r} = 0$. It is clear that they have two by two non-empty intersection. Therefore, the projective algebraic set $G(r,n)$ has an affine open covering with irreducible varieties isomorphic to $\mathbb{A}^{r(n-r)}$. Using Ex. 5, Lesson 7, we conclude that $G(r,n)$ is irreducible. Its dimension is equal to the dimension of any open subset of the open covering, $r(n-r)$. Since it is irreducible and contains open subsets isomorphic to the affine space, it is rational. $\square$

Assume $char(K) \neq 2$. In the special case $r=2$ with $n\geq 4$, using the Plücker coordinates $[\ldots, p_{ij}, \ldots]$, the equations of the Grassmannian $G(2,n)$ are of the form $p_{ij}p_{hk} - p_{ih}p_{jk} + p_{ik}p_{jh} = 0$, for any $i<j<h<k$.

Also in the case of $G(2,n)$, as for $\mathbb{P}^n\times\mathbb{P}^m$ and $V_{n,2}$, there is an interpretation in terms of matrices, that I expose here without entering in all details. Given a tensor in $\wedge^2 V$ with coordinates $[p_{ij}]$, we can consider the skew-symmetric $n\times n$ matrix whose term of position i,j is p_{ij}, with the conditions $p_{ii} = 0$ and $p_{ji} = -p_{ij}$. In this way we can construct an isomorphism between $\wedge^2 V$ and the vector space of skew-symmetric matrices of order n.

From ${}^tA = -A$, it follows $\det(A) = (-1)^n \det(A)$. If n is odd, this implies $\det(A) = 0$. If n is even, one can prove that $\det(A)$ is a square. For instance if $n=2$, and $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$, then $\det(A) = a^2$.

If $n=4$, and $P=\begin{pmatrix} 0 & p_{12} & p_{13} & p_{14} \\ -p_{12} & 0 & p_{23} & p_{24} \\ -p_{13} & -p_{23} & 0 & p_{34} \\ -p_{14} & -p_{24} & -p_{34} & 0 \end{pmatrix}$, then $\det(P)=(p_{12}p_{34}-p_{13}p_{24}+p_{14}p_{23})^2$.

In general, for a skew-symmetric matrix A of even order $2n$, one defines the **pfaffian of** A, $pf(A)$, in one of the following equivalent ways:

(i) by recursion: if $n=1$, $pf\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}=a$; if $n>1$, one defines

$$pf(A)=\Sigma_{i=2}^{2n}(-1)^i a_{1i}Pf(A_{1i}),$$

where A_{1i} is the matrix obtained from A by removing the rows and the columns of indices 1 and i. Then one verifies that $pf(A)^2=\det(A)$;

(ii) (in characteristic 0) given the matrix A, one considers the tensor $\omega=\Sigma_{i,j=1}^{2n}a_{ij}e_i\wedge e_j\in\wedge^2K^{2n}$. Then one defines the pfaffian of A as the unique constant such that $pf(A)e_1\wedge\cdots\wedge e_{2n}=\frac{1}{n!}\omega\wedge\cdots\wedge\omega$.

For a skew-symmetric matrix of odd order, one defines the pfaffian to be 0.

Proposition 1.9. *A 2-tensor $\omega\in\wedge^2V$ is totally decomposable if and only if $\omega\wedge\omega=0$.*

Proof. If ω is decomposable, the conclusion easily follows. Conversely, if $\omega=\Sigma_{i,j=1}^{2n}a_{ij}e_i\wedge e_j$ and $\omega\wedge\omega=0$, then the pfaffians of the principal minors of order 4 of the matrix A corresponding to ω are all 0, therefore from definition (ii) it follows that the pfaffians of the principal minors of all orders are 0, and also $\det(A)=0$. In conclusion A has rank 2. Then one checks that ω is the $\wedge$ product of two vectors corresponding to two linearly independent rows of A. For instance, if $a_{12}\neq 0$, then $\omega=(a_{12}e_2+\ldots+a_{1n}e_n)\wedge(-a_{12}e_1+a_{23}e_3+\ldots+a_{2n}e_n)$. □

The equations of $G(2,n)$ are the pfaffians of the principal minors of order 4 of the matrix P. They are all zero if and only if the rank of P is 2. Therefore the points of the Grassmannian $G(2,n)$, for any n, can be interpreted as (proportionality classes of) skew-symmetric matrices of order n and rank 2.

The subvarieties of the Grassmannian $\mathbb{G}(r,n)$ correspond to subvarieties of $\mathbb{P}^n$ covered by linear spaces of dimension r. Conversely, any subvariety of $\mathbb{P}^n$ covered by linear spaces of dimension r gives rise to a subvariety of the Grassmannian.

Example 1.10.

1. **Pencils of lines.** A pencil of lines in $\mathbb{P}^n$ is the set of lines passing through a fixed point O and contained in a 2-plane π such that $O\in\pi$. Assume that O has coordinates $[y_0,\ldots,y_n]$,

and fix two points $A, B \in \pi$, different from O. Let $A = [a_0, \dots, a_n]$, $B[b_0, \dots, b_n]$. Then a general line of the pencil is generated by O and by a point of coordinates $[\dots, \lambda a_i + \mu b_i, \dots]$. Therefore the Plücker coordinates of a general line of the pencil are $p_{ij} = y_i(\lambda a_j + \mu b_j) - y_j(\lambda a_i + \mu b_i) = \lambda q_{ij} + \mu q'_{ij}$, where q_{ij}, q'_{ij} are the Plücker coordinates of the lines OA and OB respectively. So the lines of the pencil are represented in the Grassmannian by the points of a line. Conversely one can check that any line contained in a Grassmannian of lines represents the lines of a pencil.

2. **Lines in a smooth quadric surface.** Let $\Sigma : x_0x_3 - x_1x_2 = \det \begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix} = 0$ be the Segre quadric in $\mathbb{P}^3$. A line of the first ruling of Σ is characterised by a constant ratio of the rows of the matrix $\begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix}$. Therefore it can be generated by two points with coordinates $[x_0, x_1, 0, 0]$, $[0, 0, x_0, x_1]$. The Plücker coordinates of such a line are $[x_0^2, 0, x_0x_2, -x_0x_2, 0, x_2^2]$. This parametrizes a conic contained in $\mathbb{G}(1,3)$. Similarly, the lines of the second ruling describe the points of another conic, indeed the coordinates are $[0, x_0^2, x_0x_1, x_0x_1, x_1^2, 0]$. These two conics are disjoint and contained in disjoint planes.

3. **Planes in** $\mathbb{G}(1,3)$. One can prove that $\mathbb{G}(1,3)$ contains two families of planes, and no linear space of dimension > 2. The planes of one family correspond to stars of lines in $\mathbb{P}^3$ (lines in $\mathbb{P}^3$ through a fixed point), while the planes of the second family correspond to the lines contained in the planes of $\mathbb{P}^3$. The geometry of the lines in $\mathbb{P}^3$ translates to give a decription of the geometry of the planes contained in $\mathbb{G}(1,3)$. Since on an algebraically closed field of characteristic $\neq 2$ two quadric hypersurfaces are projectively equivalent if and only if they have the same rank, one obtains a description of the geometry of all quadrics of maximal rank in $\mathbb{P}^5$.

Exercises 1.11. 1. Let ℓ, ℓ' two distinct lines in $\mathbb{P}^3$. Let $[p_{ij}]$ be the Plücker coordinates of ℓ and $[q_{ij}]$ those of ℓ', $0 \le i < j \le 3$. Prove that $\ell \cap \ell' \neq \emptyset$ if and only if

$$p_{01}q_{23} - p_{02}q_{13} + p_{03}q_{12} + p_{12}q_{03} - p_{13}q_{02} + p_{23}q_{01} = 0.$$

(Hint: fix points on the two lines to get the Plücker coordinates.)