$$dM(r') = 4\pi r'^2 \rho(r')\, dr'$$

$\longrightarrow$

$$\phi(r) = -\frac{G}{r}\int_0^r 4\pi r'^2 \rho(r')\, dr' - G\int_r^\infty \frac{4\pi r'^2 \rho(r')\, dr'}{r'}$$

$$= -4\pi G\left[\underbrace{\frac{1}{r}\int_0^r dr'\, r'^2 \rho(r')}_{\text{internal shells}} + \underbrace{\int_r^\infty dr'\, r'\, \rho(r')}_{\text{external shells}}\right] \quad (2.28)$$

def.
circular speed = the speed of a particle of negligible mass (= test particle) in a circular orbit at radius r
unit mass

$|F_g| = |F_{centripetal}|$ $\quad |F_g| = 1 \cdot \frac{v_c^2}{r}$

$$v_c^2 = r|F_g| = r\frac{d\phi}{dr} = \frac{GM(r)}{r} \quad (2.29)$$

circular frequency

$$\Omega \equiv \frac{v_c}{r} = \sqrt{\frac{GM(r)}{r^3}} \quad (2.30)$$

escape speed $(\cancel{T+W=0})$ $\frac{1}{2}mv_e^2 - \phi m = \phi - \cancel{\phi_\infty}$

$$v_e(r) \equiv \sqrt{2|\phi(r)|} \quad (2.31)$$

Potential energy

$$W = \int d^3\bar{x}\, \rho\, \bar{x}\, \bar{\nabla}\phi$$

in spherical coord.
$\bar{x} = r\,\hat{e}_r$
$d^3\bar{x} = 4\pi r^2\, dr$

$$W = \int 4\pi r^3 dr\, \rho\, \hat{e}_r \cdot (\bar{\nabla}\phi) =$$

$$= -4\pi\int dr\, r^3 \rho\left(\hat{e}_r \frac{GM(r)}{r^2}\hat{e}_r\right) = -4\pi G\int dr\, r\, \rho(r)\, M(r)$$

2.32

$\bar{\bar{W}}$ is diagonal
and $W_{jk} = 0 \quad j \neq k \quad W_{jk} = \frac{1}{3}|W|\delta_{jk}$ ...

0. DI POISSON

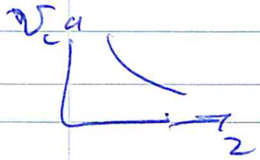
$$\nabla^2\phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = 4\pi\rho G$$

POTENZIALI X

Simple systems

a) Point-mass (Keplerian potential)

$$\phi(r) = -\frac{GM}{r} \qquad v_c = \sqrt{\frac{GM}{r}}$$



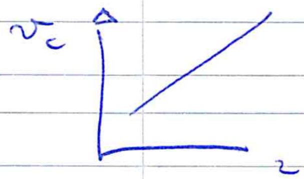
$$v_e(r) = \sqrt{\frac{2GM}{r}}$$

2.34

b) Homogeneous sphere $\rho = \text{const.}$

$$M(r) = \frac{4}{3}\pi r^3 \rho$$

$$v_c^2 = \frac{GM(r)}{r} = G\frac{\frac{4\pi}{3} r^{\cancel{3}\,2}\rho}{\cancel{r}} \rightarrow v_c = \sqrt{\frac{4\pi G\rho}{3}}\, r$$



typical time $T = \frac{2\pi r}{v_c} = \frac{2\pi}{\Omega}$

$$= \frac{2\pi \cancel{r}}{\sqrt{\frac{4\pi G\rho}{3}}\,\cancel{r}} = \sqrt{\frac{3\pi}{G\rho}} \propto \sqrt{3\pi}\,(G\rho)^{-1/2}$$

dynamical time

gravitational radius

$$r_g \equiv \frac{GM^2}{|W|} \qquad \text{homogeneous sphere } a \qquad r_g = \frac{5}{3}a$$

2.28 $\longrightarrow$ $\phi(r)$

$$r<a \quad \phi(r) = -4\pi G\left[\frac{1}{r}\int_0^r dr'\, r'^2\rho(r') + \int_r^a dr'\, r'\rho(r')\right] =$$

$$= -4\pi G\rho\left\{\frac{1}{r}\left[\frac{r'^3}{3}\right]_0^r + \left[\frac{r'^2}{2}\right]_r^a\right\} =$$

$$= -4\pi G\rho\left\{\frac{1}{\cancel{r}}\frac{r^{\cancel{3}\,2}}{3} + \frac{a^2}{2} - \frac{r^2}{2}\right\} =$$

$$= -4\pi G\rho\left\{\frac{a^2}{2} - \frac{r^2}{6}\right\} = -2\pi G\rho\left(a^2 - \frac{r^2}{3}\right)$$

$r > a$

$$\phi(r) = -4\pi G\left[\frac{1}{r}\int_0^a dr'\, r'^2 \rho(r')\right] = \frac{-4\pi G}{r}\rho\left[\frac{r^3}{3}\right]_0^a =$$

$$= -\frac{4\pi G \rho a^3}{3r}$$

↳ 1 punto massa con $M = \frac{4}{3}\pi a^3$ $\quad \phi(r) = \frac{-GM}{r}$

FOR

2.43

$$\phi(r) = \begin{cases} -2\pi G\rho\left(a^2 - \frac{1}{3}r^2\right) & (r < a) \\ -\frac{4\pi G \rho a^3}{3r} & (r > a) \end{cases}$$

is a "box" potential
NON-PHYSICAL

Real system: efforts of observers to fit observed central regions + efforts of theorist. to avoid that $M \underset{r\to\infty}{\longrightarrow} \infty$

c) Plummer model (for globular clusters!)

density ~ const center.
and then $\rho \underset{r\to\infty}{\longrightarrow} 0$

so center $\quad \phi \propto r^2 + \text{const}$
external $\quad \phi \propto r^{-1}$

$$\phi = \frac{-GM}{\sqrt{r^2+b^2}} \qquad 2.44$$

↳ Plummer scale length b
M = total mass

$$\nabla^2\phi = \frac{3GMb^2}{(r^2+b^2)^{5/2}} = 4\pi G\rho$$

eq. di Poisson

$$\boxed{\rho \underset{r\gg b}{\longrightarrow} r^{-5}}$$

f) power law density (for galaxies)

$$\boxed{3-\alpha<0} \quad \rho(r) = \rho_0 \left(\frac{r_0}{r}\right)^{\alpha} \qquad (2.58)$$

$\boxed{-\alpha<0} \to \alpha < 3$ to have a finite mass at $r \to 0$

$$M = \int dr' \, 4\pi r'^2 \rho(r') = 4\pi \int r'^2 \rho_0 \left(\frac{r_0}{r'}\right)^{\alpha} dr' =$$

$$= 4\pi \rho_0 r_0^{\alpha} \frac{r^{3-\alpha}}{3-\alpha} \qquad \left(\frac{r^{3-\alpha}}{3-\alpha} \xrightarrow[r\to 0]{} 0\right)$$

se non voglio che $M \to \infty$ $r \to \infty$ $\boxed{k<0}$

M finite $3-\alpha>0$ $\boxed{\alpha<3}$

$$v_c^2 = \frac{G M(r)}{r} = \frac{4\pi G \rho_0 r_0^{\alpha}}{3-\alpha} r^{2-\alpha} \qquad (2.61)$$

observ. $v_c \sim$ flat !
of rotation curves

$$\Longrightarrow \boxed{\alpha \sim 2}$$

$$\rho(r) = \rho_0 \left(\frac{r_0}{r}\right)^2$$

singular isothermal sphere

However M diverges at $r \to \infty$
for all $\alpha \leq 3$

but this model is useful since for the 1st Newton theorem the mass external a radius r does not affect the dynamics interior to r.

useful to fit internal regions
(see also G-L fit)

(8)

Two-power density models

$$\rho(r) = \frac{\rho_0}{(r/a)^{\alpha}\left(1+\frac{r}{a}\right)^{\beta-\alpha}} \qquad (2.64)$$

Jaffe 1983 $\boxed{\alpha=2 \;\; \beta=4}$

$$\frac{\rho_0}{\left(\frac{r}{a}\right)^2\left(1+\frac{r}{a}\right)^2} \xrightarrow[r\to 0]{} r^{-2}(r+a)^{-2}$$

$\boxed{\alpha=1 \;\; \beta=4}$ Hernquist 1990

$$\frac{\rho_0}{\left(\frac{r}{a}\right)\left(1+\frac{r}{a}\right)^3} \xrightarrow[r\to 0]{} r^{-1}(r+a)^{-3}$$

Navarro Frenk White 1995 $\boxed{\alpha=1 \;\; \beta=3}$

NFW model

$$\frac{\rho_0}{\left(\frac{r}{a}\right)\left(1+\frac{r}{a}\right)^2} \xrightarrow[r\to 0]{} r^{-1}(r+a)^{-2}$$

From simulations → ρ_0 and a are correlated

FOR ALL $\rho \xrightarrow[r\to 0]{} \infty$ NO GOOD!

r_{200} where the mean density is $200\,\rho_c$

$M_{200} = 200\,\rho_c \frac{4\pi}{3} r_{200}^3$

$c = \frac{r_{200}}{a}$

$M(r) = \int_0^r dr' 4\pi r'^2 \rho(r') =$

$= \ldots$

For J e H models

$M \xrightarrow[r\to\infty]{}$ finite value

For NFW

$M \xrightarrow[r\to\infty]{} \infty$

Sersic - 3D (Einasto model)

$$\rho(r) = \rho_0 \exp\left[-\left(\frac{r}{a}\right)^{1/m}\right] \qquad m \sim 6$$

Formally better than NFW
Einasto 1969

$$\rho \xrightarrow[r\to 0]{} \text{finite}$$

$$M \xrightarrow[r\to\infty]{} \text{finite}$$

Figure of Gary Mamon COURSE