$\longleftrightarrow$ POLYTROPIC GAS

$$p = K \rho^{\gamma} \qquad (1E-27)$$

eq. of hydrostatic equilibrium for a self gravitating sphere of polytropic gas

Euler eq. $\underbrace{\frac{\partial \bar{v}}{\partial t} + (\bar{\nabla} \bar{v}) \bar{v}}_{=0} = \underbrace{\frac{1}{\rho}\frac{dp}{dr} - \frac{d\phi}{dr} = 0}_{\text{spherical system}}$

$\rho \to \phi$ selfgravitating

$$\frac{dp}{dr} = -\rho \frac{d\phi}{dr} \qquad \boxed{4-109a} \qquad \phi = -\Psi + \phi_0$$

$$K \gamma \rho^{\gamma-1} \frac{d\rho}{dr} = \rho \frac{d\Psi}{dr}$$

$$K \gamma \rho^{\gamma-2} \frac{d\rho}{dr} = \frac{d\Psi}{dr} \qquad \boxed{4-109b}$$

If $\Psi = 0$ on the edge of the system where $\rho = 0$ and integrate

$$\int_0^{\rho} \rho'^{\gamma-2} \, d\rho' = \frac{1}{K\gamma} \int_0^{\Psi} d\Psi'$$

$$\rho^{\gamma-1} = \frac{\gamma-1}{K\gamma} \Psi$$

$$\rho = \left(\frac{\gamma-1}{K\gamma}\right)^{\frac{1}{\gamma-1}} \Psi^{\frac{1}{\gamma-1}} \qquad \boxed{4-110}$$

cf with 4-107e of stellar poly

$c_n =$ $\qquad n = \frac{1}{\gamma-1} \qquad \gamma = 1 + \frac{1}{n}$ $\qquad \boxed{n > \frac{1}{2} \Rightarrow \gamma < 3}$

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Solutions of Emden eq, 4-108c

i) $n=1$ $\rightarrow$ Helmholtz eq.

ii) $n=5$
$$\varphi = \frac{1}{\sqrt{1+\frac{1}{3}s^2}}$$

In fact...

$$\frac{1}{s^2}\frac{d}{ds}\left(s^2\frac{d\varphi}{ds}\right) = -\frac{1}{3s^2}\frac{d}{ds}\left[\frac{s^3}{(1+\frac{1}{3}s^2)^{3/2}}\right] = \dots =$$
$$= \dots -\varphi^5$$

According to 4-107e ~~ρ~~

$$\rho = C_5 \psi^5 =$$
$$= C_5 \psi_0^5 \varphi^5 = \frac{C_5 \psi_0^5}{(1+\frac{1}{3}s^2)^{5/2}}$$

$$\rho \xrightarrow[r\to\infty]{} s^{-5} \quad \text{that is} \rightarrow r^{-5}$$

Mass is finite! $M_\infty = \frac{1}{G}\left(r^2\frac{d\phi}{dr}\right) \dots = \frac{\sqrt{3}\psi_0 b}{G}$
$r\to\infty$

Plummer $\rho \rightarrow n=5$ good for globular cluster!

$n<5$ $\rho \to 0$ at finite ~~at any~~ radius

$n=5$ $\rho \underset{r\to\infty}{\sim} r^{-5}$ $M_{r\to\infty}$ is finite

$n>5$ ρ density falls off so slowly
$M \xrightarrow[r\to\infty]{} \infty$

$\boxed{n\to\infty}$ isothermal sphere

(b) The ISOTHERMAL SPHERE

$n \to \infty$ polytropic stellar systems

$n = \frac{1}{\gamma - 1}$ gas poly with $\gamma \sim 1 \to p = K\rho$ isothermal gas!

Lane-Emden eq. is not useful
but we use gas

$\frac{dp}{dr} = -\rho \frac{d\phi}{dr}$ $\xrightarrow[\text{isothermal gas}]{+}$ $p = \frac{k_B T \rho}{m}$

hydrostatic eq.

$$\frac{dp}{dr} = \underbrace{\frac{k_B T}{m}}_{\text{mass particle}} \frac{d\rho}{dr} = -\rho \frac{G M(r)}{r^2} \qquad 4.115\,a$$

$4.115 \times \frac{r^2 m}{\rho k_B T}$ and $\frac{d}{dr}$

$$\frac{r^2}{\rho} \frac{d\rho}{dr} = -\frac{G m\, \cancel{r^2} M}{k_B T\, \cancel{r^2}}$$

$$r^2 \frac{d \ln \rho}{dr} = -\frac{G m}{k_B T} \int \rho\, 4\pi r^2 dr$$

$$\frac{d}{dr}\left(r^2 \frac{d \ln \rho}{dr}\right) = -\frac{G m}{k_B T} 4\pi r^2 \rho \qquad \boxed{4\text{-}115\ b} \ \boxed{\text{gas}}$$

Assume $f(\mathcal{E}) = \frac{\rho_1}{(2\pi \sigma^2)^{3/2}} e^{\mathcal{E}/\sigma^2} = \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} e^{\frac{\Psi - \frac{1}{2}v^2}{\sigma^2}}$ $\boxed{4.116}$

$\rho = \int f(\mathcal{E})\, d^3v$

$$\rho = \rho_1 e^{\frac{\Psi}{\sigma^2}} \int_0^\infty \frac{4\pi}{(2\pi\sigma^2)^{3/2}} e^{\frac{-\frac{1}{2}v^2}{\sigma^2}} v^2 dv =$$

$$= \rho_1 e^{\frac{\Psi}{\sigma^2}} \underbrace{\int_0^\infty \frac{4}{\sqrt{\pi}} \frac{1}{(2\sigma^2)^{3/2}} e^{\frac{-\frac{1}{2}v^2}{\sigma^2}} v^2 dv}_{\text{è Maxwelliana } \int = 1}$$

$$\rho = \rho_1 \, e^{\frac{\psi}{\sigma^2}} \qquad 4.117$$

Poisson eq. 4-104 →

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\psi}{dr}\right) = -4\pi G \rho$$

$$\psi = \left(\ln \frac{\rho}{\rho_1}\right) \cdot \sigma^2$$

$$\frac{d}{dr}\left(r^2 \sigma^2 \frac{d\ln\rho - \cancel{d\ln\rho_1}}{dr} \right) = -4\pi G \rho r^2$$

$$\frac{d}{dr}\left(r^2 \frac{d\ln\rho}{dr}\right) = \frac{-4\pi G}{\sigma^2} r^2 \rho$$

stella system

$\boxed{4-119e}$

cf. con 4-115b

$$\cancel{\frac{d}{dr}\left(r^2 \frac{d\ln\rho}{dr}\right)}$$

1D vedremo dopo

$$\boxed{\sigma^2 = \frac{K_B T}{m}} \rightarrow \text{gas}$$

velocity dispersion

structure of ~~an~~ isothermal [collision] self-gravitating sphere of gas is = structure of a [collisionless] system of stars described by $f(\mathcal{E})$

the distribution of velocities is a Maxwellian distribution

$$F(v) = N e^{-\frac{1}{2}\frac{v^2}{\sigma^2}} \qquad F(v) = \frac{N f}{v}$$

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Mean square speed of the stars at a point in the isothermal sphere

$$\overline{v^2} = \frac{\int_{v}^{\infty} v^2 \cdot f \, d^3 v}{\int_{v}^{\infty} f \, d^3 v} =$$

$$= \frac{\int_0^\infty v^2 \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} e^{\frac{\Psi - \frac{1}{2}v^2}{\sigma^2}} \cancel{4\pi} v^2 dv}{\int_0^\infty \frac{\rho_1}{\cdots} e^{\frac{\Psi - \frac{1}{2}v^2}{\sigma^2}} \cancel{4\pi} v^2 dv} = \qquad \frac{1}{2} v^2 = x^2$$

$$= 2\sigma^2 \frac{\int_0^\infty e^{-x^2} x^4 dx \rightarrow \frac{3\sqrt{\pi}}{8}}{\int_0^\infty e^{-x^2} x^2 dx \rightarrow \frac{\sqrt{\pi}}{4}} = \cancel{2}\sigma^2 \frac{3}{\cancel{8}_2} \cancel{\sqrt{\pi}} \cdot \frac{\cancel{4}}{\cancel{\sqrt{\pi}}} = 3\sigma^2$$

$$\overline{v_r^2} = \overline{v_\theta^2} = \overline{v_\phi^2} \longrightarrow \neq \cancel{\ \ }^2$$

$$\overline{v^2} = 3\overline{v_r^2} = 3\sigma^2 \longrightarrow v_r^2 = \sigma^2 \rightarrow \text{1D}$$

4-119 a → SOLUTION?

$$\boxed{\rho = C\, r^{-b}} \quad 4.123$$

yes! $\frac{d}{dr}\left(r^2 \frac{d \ln C r^{-b}}{dr}\right) = \frac{-4\pi G}{\sigma^2} r^2 C r^{-b}$

$\frac{d}{dr}\left(r^2 \frac{-bC\, r^{-b-1}}{\cancel{C r^{-b}}}\right) = \frac{-4\pi G}{\sigma^2} C r^{2-b}$

SINGULAR ISOTHERMAL SPHERE

$\frac{d}{dr}(-b) = \dots$

$$\boxed{\rho = \frac{\sigma^2}{2\pi G} r^{-2}}$$

$-b = -\frac{4\pi G}{\sigma^2} C r^{2-b}$

$\rho \xrightarrow[r \to 0]{} \infty$ ← PROBLEM 1

x to be true

$2 - b = 0 \rightarrow \boxed{b = 2}$

TO solve this

$-2 = -\frac{4\pi G}{\sigma^2} C \quad \boxed{C = \frac{\sigma^2}{2\pi G}}$

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new dimensionless variable

$\tilde{\rho}$ $\tilde{r}$

$$\tilde{\rho} \equiv \frac{\rho}{\rho_0} \quad (\rho \text{ at } r=0) \qquad \tilde{r} \equiv \frac{r}{r_0} \qquad 4.124e$$

r_0 → king radius

4-123 →

$$\tilde{\rho}\rho_0 = \frac{\sigma^2}{2\pi G r_0^2 \tilde{r}^2}$$

$$\tilde{\rho} = \underbrace{\frac{\sigma^2}{\rho_0 2\pi G r_0^2} \cdot \frac{9}{2}}_{=1} \cdot \frac{2}{9} \frac{1}{\tilde{r}^2}$$

$$\rightarrow \quad r_0 \equiv \sqrt{\frac{9\sigma^2}{4\pi G\rho_0}}$$

is the projected radius
at which the
projected density
is ~ half
of its central value

4-123

$$\tilde{\rho} = \frac{2}{9} \tilde{r}^{-2}$$

4-118 di Poisson →

$$d \ln(\tilde{\rho}\rho_0) = d(\ln\tilde{\rho} + \ln\rho_0)$$

$$\frac{d}{r_0 d\tilde{r}} \left(r_0^2 \tilde{r}^2 \frac{d \ln \tilde{\rho}}{r_0 d\tilde{r}} \right) = \frac{-4\pi G \rho_0 \tilde{\rho} \tilde{r}^2 r_0^2}{\sigma^2}$$

$$\frac{d}{d\tilde{r}} \left[\tilde{r}^2 \frac{d \ln \tilde{\rho}}{d\tilde{r}} \right] = -9 \tilde{r}^2 \tilde{\rho} \qquad 4\text{-}125e$$

Ct. Isothermal sphere

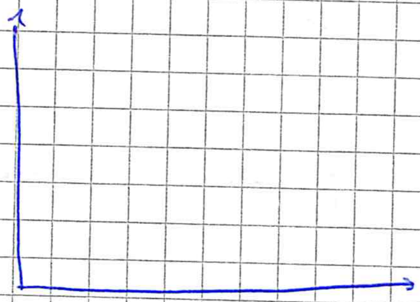

Fig 4.7

ch. isothermal or Hubble law

Abel

$$\Sigma(R) = \int_{-\infty}^{+\infty} \rho(r)\, dz = 2\int_R^{\infty} \frac{\rho(r)\, r\, dr}{\sqrt{r^2-R^2}} \ldots$$

$z=\sqrt{r^2-R^2}$

$$\Sigma(R) = \frac{\sigma^2}{2GR} = \frac{2}{9}\pi \rho_0 r_0 \left(\frac{r_0}{R}\right) \qquad 4.126$$

$$\Sigma(R) \propto R^{-1}$$

circular speed $m\frac{v_c^2}{r} = G\frac{Mm}{r^2}$

$M=\int \rho\, 4\pi r^2 dr$

$$v_c^2(r) = \frac{GM(r)}{r} \propto \frac{r}{r} \sim \text{const}$$

$$= \int \frac{\sigma^2 4\pi r^2}{2\pi G r^2}\, dr \propto r$$

OK con of rotation VELOC. GALS! * in SPi v = const external

At $\tilde r \le 2$ ($r \le 2r_0$) approx. to $\tilde\rho(\tilde r)$ is the MODIFIED HUBBLE LAW

$$\tilde\rho(\tilde r) \sim \tilde\rho_h(\tilde r) \equiv \frac{1}{(1+\tilde r^2)^{3/2}} \underset{r\to\infty}{\propto} \tilde r^{-3} \qquad 4\text{-}128$$

internal ~ isot

external [illegible] r^{-3} as Hubble law. For [illegible]

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CORE FITTING METHOD

many ellipticals are well fitted by an isothermal sphere out to few core radii

obs. photometry + fit $\longrightarrow$ King radius r_0

obs. measure $\sigma^2_{LOS} \rightarrow \sigma^2$

4.124b

$$r_0 = \sqrt{\frac{9\sigma^2}{4\pi G\rho_0}} \quad + r_0 + \sigma^2 \longrightarrow \boxed{\rho_0}$$

obs. central luminosity density j_0

$$\longrightarrow \Upsilon = \frac{M}{L} = \frac{\rho_0}{j_0}$$ MASS TO LIGHT RATIO

PROBLEM of isoth. sphere $\quad M \rightarrow \infty$

To solve this

(C) lowered isothermal models

we look for something ~ isothermal sphere at small radii where $\mathcal{E}$ of stars are large

less dense isoth. at large radii (when $\mathcal{E}$ small values)

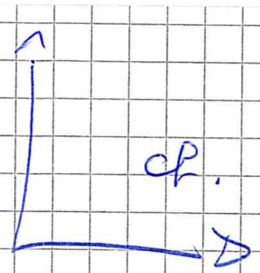
4-116

$$f_K(\mathcal{E}) = \begin{cases} \rho_1 (2\pi\sigma^2)^{-3/2} \left(e^{\mathcal{E}/\sigma^2} - 1\right) & \mathcal{E} > 0 \\ 0 & \mathcal{E} \leq 0 \end{cases}$$ $\boxed{4.130}$

KING MODELS

Tronca limit energy. $\mathcal{E}/\sigma^2$ important for small $\mathcal{E}$!

FIG 4.9



cf. King model

Sometime tidal radius r_t
which defines concentration

$$c = \lg\left(\frac{r_t}{r_0}\right)$$

r_0 small $\rightarrow$ concentration high

Wilson model (MAMON)
better for global or cluster

$$f(E) = \text{cost}\left[e^{\left(-\frac{E}{\sigma_v^2}\right)} - 1 + \frac{E}{\sigma_v^2}\right]$$

$$f(\varepsilon) = \text{cost}\left(e^{\frac{\varepsilon}{\sigma^2}} - 1 - \frac{\varepsilon}{\sigma^2}\right)$$

Systems with anisotropic dispersion tensors

Michie

$$f(E, J) = \text{cost}\left[e^{-\frac{E}{\sigma_v^2}} - 1\right] e^{-\frac{J^2}{2 r_a^2 \sigma_v^2}}$$

(at the center. King (E is important)
velocity isotropy

external region J is important
velocity anisotropy

Osipkov-Merritt

$$f(E, J) = f\left(E + \frac{J^2}{2 r_a^2}\right)$$

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